14.10.2017

$\boxed{\begin{array}{c}Q4\\ \S\S7.9\end{array}}$ Compute the integral $\int_{-\infty}^{\infty} \frac{x^2 \exp(px)}{1+\exp(x)}\, dx \quad 0<p<1$ using the method of contour integration.

We shall relate this integral to $\int_{-\infty}^{\infty} \frac{x \exp(px)}{1+\exp(x)}\, dx$ which has already been evaluated in Q[3]. Alternately the steps below could be used to derive a recurrence relation for $I_n = \int_{-\infty}^{\infty} \frac{x^n \exp(px)}{1+\exp(x)}\, dx$.

We set up the integral of

$f(z) = \frac{z^2 \exp(pz)}{1+\exp(z)}$ around the contour Γ of Fig 1. As $L \to \infty$ the integrals of $f(z)$ along BC and DA tend to zero.

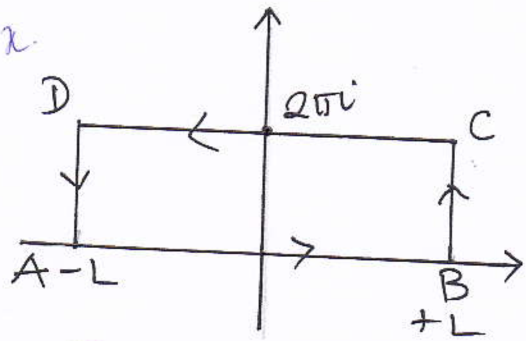

Fig1 Contour Γ

In this limit

$$\oint_\Gamma f(z)\, dz = \int_{-\infty}^{\infty} \frac{x^2 \exp(px)}{1+\exp(x)}\, dx - \int_{-\infty}^{\infty} \frac{(x+2\pi i)^2 \exp(2\pi i p) \times \exp(px)}{1+\exp(x)}$$

$$= (1-\exp(2\pi i p)) \int_{-\infty}^{\infty} \frac{x^2 \exp(px)}{1+\exp(x)}\, dx$$

$$- 4\pi i e^{2\pi i p} \int_{-\infty}^{\infty} \frac{x e^{px}}{1+e^x}\, dx + (4\pi^2) e^{2\pi i p} \int_{-\infty}^{\infty} \frac{\exp(px)}{1+\exp(x)}\, dx$$

Compute residue of $f(z)$ at $z = \pi i$

$$\oint_\Gamma f(z)dz = 2\pi i \operatorname{Res}\left\{\frac{z^2 e^{pz}}{1+e^z}\right\}_{z=\pi i}$$

$$= 2\pi i \lim_{z\to\pi i}\left(\frac{z^2 e^{pz}}{1+e^z}\right)(z-\pi i)$$

$$= 2\pi i \left(z^2 e^{pz}\right)_{z=\pi i} \lim_{z\to\pi i}\left(\frac{z-\pi i}{1+e^z}\right)$$

$$= 2\pi i\,(\pi^2 i^2)\, e^{\pi i p} \times (-1)$$

$$= 2\pi^3 i\, e^{\pi i p}$$

Using notation

$$I = \int_{-\infty}^{\infty} \frac{x^n e^{px}}{1+e^x}\,dx$$

We get

$$(1-e^{2\pi i p})\, I_2 - 4\pi i e^{2\pi i p} I_1 + (4\pi)^2 e^{2\pi i p} I_0 = (2\pi)^3 i e^{\pi p i}$$

multiply by $i e^{-i\pi p}$ to get

$$2\sin\pi p\, I_2 + 4\pi\; e^{i\pi p} I_1 + i(4\pi)^2 e^{i\pi p} I_0 = -(2\pi)^3$$

Equating real and imaginary parts,

$$2\sin\pi p\, I_2 + 4\pi \cos\pi p\, I_1 - 4\pi^2 I_0 \sin\pi p = 0.$$

$$4\pi I_1 \sin\pi p + 4\pi^2 \cos\pi p\, I_0 = -2\pi^3$$

We need to do one of the two integrals I_0 or I_1. These two integrals have been evaluated in Q[3]. Borrowing the answers from solution of Q[3]

$$I_0 = \pi \operatorname{cosec} p\pi \qquad I_1 = -\pi^2 \cot\pi p \operatorname{cosec} p\pi$$

This gives $I_2 = -\pi^3 \operatorname{cosec}(p\pi) - 2\pi \cot p\pi\, I_1 + 2\pi^2 I_0$

$$I_2 = -\pi^3 \operatorname{cosec}(\pi p) - 2\pi \cot(p\pi)\left(-\pi^2 \cot(p\pi)\operatorname{cosec}(p\pi)\right)$$
$$+ 2\pi^2\left(\pi \operatorname{cosec}(p\pi)\right).$$

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$$I_2 = -\pi^3 \operatorname{cosec}(p\pi) + 2\pi^3 \cot^2(p\pi)\operatorname{cosec}(p\pi) + 2\pi^3 \operatorname{cosec}(p\pi)$$

$$= \pi^3 \operatorname{cosec}(p\pi)\left(1 + 2\cot^2(p\pi)\right)$$

$$= \pi^3 \operatorname{cosec}(p\pi)\left(-1 + 2\operatorname{cosec}^2(\pi p)\right)$$

$$= \pi^3 \operatorname{cosec}^3(p\pi)\left(-\sin^2 p\pi + 2\right)$$

$$= \frac{\pi^3}{2}\operatorname{cosec}^3(p\pi)\left(3 + \cos(2p\pi)\right)$$