p1/Q[3]/§§7.9

Q[3] §§7.9 Integrate $\int_{-\infty}^{\infty} \frac{x \exp(px)}{1+\exp(x)}\,dx \quad 0<p<1$ using the method of contour integration.

Use rectangular contour Γ of Fig 1 and integrate $f(z) = \frac{z \exp(pz)}{1+\exp(z)}$. Then we have

$$\oint_{\Gamma} f(z)\,dz = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} f(z)\,dz$$

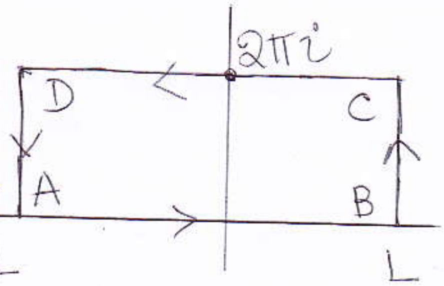

Fig 1. Contour Γ

The Darboux theorem implies that $\int_{BC} f(z)\,dz$ and $\int_{DA} f(z)\,dz$ tend to zero as $L \to \infty$. In this limit

$$\oint_{\Gamma} f(z)\,dz = \lim_{L\to\infty} \int_{AB} f(z)\,dz - \int_{DC} f(z)\,dz$$

$$= \int_{-\infty}^{\infty} \frac{x \exp(px)}{1+\exp(x)}\,dx - \int_{-\infty}^{\infty} (x+2\pi i) e^{2\pi i p} \frac{e^{xp}}{1+\exp(x)}\,dx$$

$$= (1-e^{2\pi i p})\, I - 2\pi i\, I_2$$

For DC
$z = x + 2\pi i$
$dz = dx \quad -L \le x \le L$

where $I_1 = \int_{-\infty}^{\infty} \frac{x \exp(px)}{1+\exp(x)}\,dx$

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{px}\,dx}{1+\exp(x)} \qquad \text{---}(1)$$

The integral $\oint_{\Gamma} f(z)\,dz$ is computed using residue theorem

$f(z)$ has poles at $z = (2n+1)\pi i$. Of these only one pole at $z = \pi i$ is enclosed inside the contour Γ.

$$\operatorname{Res}\{f(z)\}_{z=\pi i} = \lim_{z \to \pi i} (z - i\pi) \frac{z e^{pz}}{1+\exp(z)}$$

$\lim(f \times g) = \lim f \times \lim g$ (when both limits exist)

$$= \pi i\, e^{\pi i p} \lim_{z \to i\pi} \frac{z - \pi i}{1+\exp(z)}$$

use L'Hopital's rule

$$= \pi i\, e^{i\pi p} \times \frac{1}{(-1)}$$

$$= -\pi i \exp(i\pi p)$$

$\therefore$ Eq(1) gives

$$(1 - e^{2\pi i p}) I_1 - 2\pi i I_2 = (2\pi i) \times (-\pi i) \exp(i\pi p)$$

$$(e^{-i\pi p} - e^{i\pi p}) I_1 - 2\pi i\, e^{i\pi p} I_2 = 2\pi^2$$

$$-2i \sin \pi p\, I_1 - 2\pi i (\cos \pi p + i \sin \pi p) I_2 = 2\pi^2$$

Equate real and imaginary parts to get

$$2\pi \sin p\pi\, I_2 = 2\pi^2 \Rightarrow I_2 = \pi \operatorname{cosec}(p\pi)$$

$$\sin \pi p\, I_1 + \pi \cos \pi p\, I_2 = 0$$

$$I_1 = -\pi^2 \cot \pi p \operatorname{cosec}(p\pi)$$