23.9.2017

P1/Q[12]/§§7.7

§§7.7
Q[12] Show that $\displaystyle\int_0^\infty \frac{x^2 \log x}{(a^2+b^2x^2)(1+x^2)} = \frac{a\pi \log(b/a)}{2b(b^2-a^2)}$

Let $f(z) = \dfrac{{}^2\mathrm{Log}\, z}{(a^2+b^2z^2)(1+z^2)}$

and use principal branch of Log z.
Consider

$$\oint_\Gamma f(z)\,dz = \int_{OB} f(z)\,dz + \int_{BCA} f(z)\,dz + \int_{AO} f(z)\,dz$$

In the limit $R \to \infty$ the integral along BCA tends to zero. Therefore

$$\oint_\Gamma f(z)\,dz = \int_{OB} f(z)\,dz - \int_{OA} f(z)\,dz$$

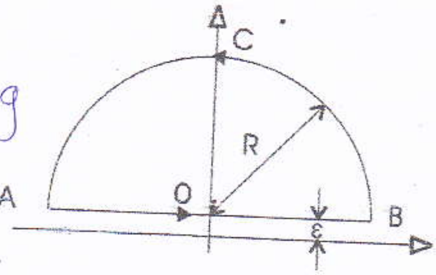


Fig. 1 Contour Γ

along OA $\quad 0<r<R$

$z=-r,\ dz=-dr,$

$$f(z) = \frac{r^2(\ln r + i\pi)}{(a^2+b^2r^2)(1+r^2)}$$

along OB, $r<r<R$

$z=r,\ dz=dr$

$$f(z) = \frac{r^2 \ln r}{(a^2+b^2r^2)(1+r^2)}$$

Hence we have

$$\oint_\Gamma f(z)\,dz = 2\int_0^\infty \frac{r^2 \ln r}{(a^2+b^2r^2)(1+r^2)} + i\pi\int_0^\infty \frac{dr}{(a^2+b^2r^2)(1+r^2)}$$

Only poles at $z=i$ and $z=i\,a/b$ contribute to the integral $\oint_\Gamma f(z)\,dz$. (We assume $b/a>0$ other cases are left to the reader.)

Compute the residues of $f(z)$ P2/Q[12]/§§7.7

$$\mathrm{Res}\{f(z)\}_{z=i} = \lim_{z\to i} \frac{(z-i)\, z^2 \,\mathrm{Log}\, z}{(z^2+1)(a^2+b^2z^2)}$$

$$= \frac{(-1)(i\pi/2)}{2i(a^2-b^2)} = \frac{\pi}{4(b^2-a^2)}$$

$$\mathrm{Res}\{f(z)\}_{z=ia/b} = \lim_{z\to \frac{ia}{b}} \frac{(z-ia/b)\, z^2 \,\mathrm{Log}\, z}{(z^2+1)(b^2)(z^2+a^2/b^2)}$$

$$= \frac{(-a^2/b^2)(\ln(a/b)+i\pi/2)}{(1-a^2/b^2)\, b^2\,(2ia/b)}$$

{We assume $a/b>0$. other cases reader should work out.

$$= \frac{a}{2b}\,\frac{(\ln(a/b)+i\pi/2)}{(b^2-a^2)} \qquad \text{---- (2)}$$

$$\therefore \oint_\Gamma \frac{z^2 \ln z\, dz}{(a^2+b^2z^2)(1+z^2)} = 2\pi i \times \left( \frac{\pi}{4}\frac{1}{(b^2-a^2)} + \frac{ai}{2b}\,\frac{\ln(a/b)+i\pi/2}{b^2-a^2} \right)$$

$$= \frac{\pi a}{b}\,\frac{\ln(b/a)}{(b^2-a^2)} + \frac{(2\pi i)}{4}\left( \frac{\pi}{(b^2-a^2)} + \frac{a}{b}\frac{\pi}{(b^2-a^2)} \right) \qquad \text{---- (3)}$$

Equating real and imaginary parts we get

$$\int_0^\infty \frac{x^2 \log x\, dx}{(x^2+1)(a^2+b^2x^2)} = \frac{\pi a}{2b}\,\frac{\ln(b/a)}{(b^2-a^2)}$$

and

$$\int_0^\infty \frac{x^2\, dx}{(x^2+1)(a^2+b^2x^2)} = \frac{\pi}{2b(a+b)}$$