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§§7.7 Q[11] Using the method of contour integration prove that

$$\int_0^{\infty} \frac{\operatorname{Log} x \, dx}{x^2+2ax\cos\alpha+a^2} = \frac{\alpha \operatorname{Log} a}{a \sin\alpha}$$

This integral requires two circles contour of Fig 2. The reader may check that Γ of Fig 1 does not yield a proof of given result.

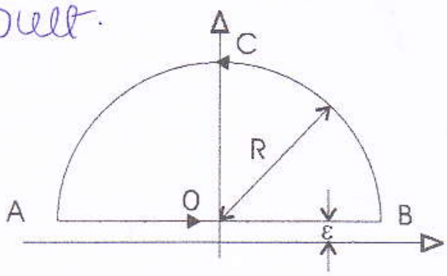

Fig. 1 Contour Γ

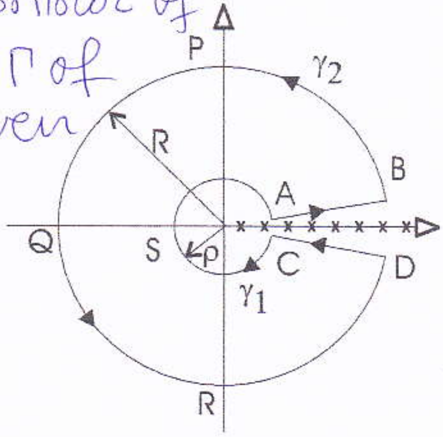

Fig. 2 Two Circles Contour

Let $\operatorname{Log} z = \ln r + i\theta$, $0<\theta<2\pi$. and we integrate

$$f(z) = \frac{(\operatorname{Log} z)^2}{z^2+2az\cos\alpha+a^2}$$ around closed contour of Fig 2.

In the limit $\rho \to 0$, $R\to\infty$, the integral of $f(z)$ along the circles γ_1, γ_2 tends to zero. This gives

$$\oint_C f(z)\,dz = \int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz$$

$$= \int_0^{\infty} \operatorname{Disc}(f(z))\,dz$$

where $\operatorname{Disc}(f(z))$ is the discontinuity across the branch cut $\theta=0$. For sake of definiteness we will assume $0<\alpha<\pi$.

$$\text{Disc}\{f(z)\} = f(x+i\epsilon) - f(x-i\epsilon)$$

$$= \frac{(\ln x)^2 - (\ln x + 2\pi i)^2}{x^2 + 2ax\cos\alpha + a^2}$$

$$= \frac{-4\pi i \ln x + 4\pi^2}{x^2 + 2ax\cos\alpha + a^2}$$

$$\therefore \oint_C f(z)\,dz = \int_0^\infty \frac{-4\pi i \ln x + 4\pi^2}{x^2 + 2ax\cos\alpha + a^2} \qquad \text{---- (1)}$$

The function $f(z)$ has two simple poles at locations given by roots of

$$z^2 + 2az\cos\alpha + a^2$$

$$z = \frac{-2a\cos\alpha \pm \sqrt{4a^2\cos^2\alpha - 4a^2}}{2}$$

$$= -a\cos\alpha \pm i a\sin\alpha = -ae^{i\alpha}, -ae^{-i\alpha}$$

For definiteness assume $0 < \alpha < \pi$ and write

$$z_1 = a e^{i(\pi+\alpha)} \qquad z_2 = a e^{i(\pi-\alpha)}$$

Note for $\text{Log}\, z_1$ and $\text{Log}\, z_2$, $\arg z_1$, $\arg z_2$ must lie in range $(0, 2\pi)$.

Computing residues of $f(z)$ at $z = z_1$

$$\text{Res}\{f(z)\}_{z=z_1} = \lim_{z\to z_1} \frac{(\ln a + i\alpha)^2}{(z+ae^{i\alpha})(z+ae^{-i\alpha})} \times (z + ae^{i\alpha})$$

$$= \frac{(\ln a + i\alpha)^2}{-ae^{i\alpha} + ae^{-i\alpha}} = \frac{(\ln a + i\alpha)^2}{-2ai\sin\alpha} \qquad \text{---- (2)}$$

$$\text{Res}\{f(z)\}_{z=z_2} = \lim \frac{(\ln a + i\alpha)^2 (z + ae^{-i\alpha})}{(z+ae^{i\alpha})(z+ae^{-i\alpha})}$$

$$= \frac{(\ln a + i\alpha)^2}{2ai\sin\alpha} \qquad \text{---- (3)}$$

Using the residue theorem, LHS of Eq (1) becomes

$$(2\pi i)\times\left[\frac{(\ln a+i(\pi+\alpha))^2}{-2ai\sin\alpha}+\frac{(\ln a+i(\pi-\alpha))^2}{2ai\sin\alpha}\right]$$

$$=\left(\frac{\pi}{a\sin\alpha}\right)\left((\ln a+i(\pi-\alpha))^2-(\ln a+i(\pi+\alpha))^2\right)$$

$$=\frac{\pi}{a\sin\alpha}(2\ln a+2i\pi)(-2i\alpha)$$

$$=\frac{-4\pi}{a\sin\alpha}((\alpha\ln a)i-\pi\alpha) \quad \text{-----} \quad (4)$$

Equating real and imaginary parts of (1) and (4) we get

$$\int_0^\infty \frac{\ln x\, dx}{(x^2+2ax\cos\alpha+a^2)}=\frac{\alpha\log a}{a\sin\alpha}$$

$$\int_0^\infty \frac{dx}{x^2+2ax\cos\alpha+a^2}=\frac{\alpha}{a\sin\alpha}$$