18.9.2017

Q[8] §§7.7 Compute $\int_0^\infty \frac{x^2 \log x}{(x^2+1)^2}\,dx$ using the method of contour integration.

We take the principal branch of $\operatorname{Log} z$

$$\operatorname{Log} z = \ln r + i\theta, \quad -\pi < \theta < \pi,$$

and set up integral of

$\frac{z^2 \operatorname{Log} z}{(z^2+1)^2}$ around the contour Γ of Fig 1. We will be taking limits $R \to \infty$, $\varepsilon \to 0$. In this limit

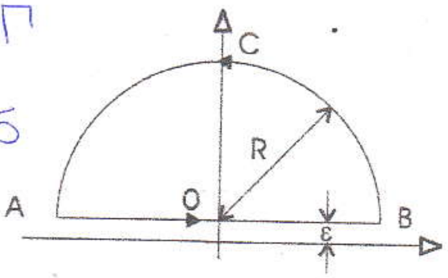

Fig. 1 Contour Γ

$$\int_{BCA} \frac{z^2 \operatorname{Log} z}{(z^2+1)^2}\,dz \to 0$$

Therefore

$$\oint_\Gamma \frac{z^2 \operatorname{Log} z}{(z^2+1)^2}\,dz = \int_{AO} \frac{z^2 \operatorname{Log} z}{(z^2+1)^2}\,dz + \int_{OB} \frac{z^2 \operatorname{Log} z}{(z^2+1)^2}$$

$$= -\int_{OA} \frac{z^2 \operatorname{Log} z\,dz}{(z^2+1)^2} + \int_{OB} \frac{z^2 \operatorname{Log} z}{(z^2+1)^2}\,dz \quad ---(1)$$

Along OA $z = re^{i\pi}$, $0 < r < R$

$dz = -dr$

$\operatorname{Log} z = \ln r + i\pi$

Along OB $z = r$, $0 < r < R$

$dz = dr$

$\operatorname{Log} z = \ln r$

Therefore

$$\oint \frac{z^2 \operatorname{Log} z}{(z^2+1)^2}\,dz = \int_0^\infty \frac{r^2 \ln r}{(r^2+1)^2}\,dr + \int_0^\infty \frac{i\pi \cdot r^2\,dr}{(r^2+1)^2}$$

$$+ \int_0^\infty \frac{r^2 \ln r}{(r^2+1)^2}\,dr \quad ---(2)$$

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$$\text{L.H.S.} = (2\pi i)\,\text{Res}\left\{\frac{z^2\,\text{Log}\,z}{(z^2+1)^2}\right\}_{z=i}$$

$$= (2\pi i)\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{z^2\,\text{Log}\,z}{(z^2+1)^2}\right]$$

$$= 2\pi i\,\frac{d}{dz}\,\frac{z^2\,\text{Log}\,z}{(z+i)^2}\Big|_{z=i}$$

$$= 2\pi i\left(\frac{z^2\,\text{Log}\,z}{(z+i)^3}\times(-2)+\frac{2z\,\text{Log}\,z+z}{(z+i)^2}\right)_{z=i}$$

$$= 2\pi i\left(\frac{+\cancel{2}\left(\frac{i\pi}{\cancel{2}}\right)}{(-8i)}+\frac{2(i)(i\pi/2)+i}{-4}\right) \qquad \text{Log}\,i = \frac{i\pi}{2}$$

$$= -\frac{\pi^2}{4}i + 2\pi i\left(\frac{-\pi+i}{-4}\right)$$

$$= -\frac{\pi^2}{4}i + i\frac{\pi^2}{2}+\frac{\pi}{2} = \frac{i\pi^2}{4}+\frac{\pi}{2}$$

Substituting in (2)

$$2\int_0^\infty \frac{x^2\log x}{(x^2+1)^2}dx + i\pi\int_0^\infty \frac{r^2dr}{(r^2+1)^2} = \frac{\pi}{2}+i\frac{\pi^2}{4}$$

We get

$$\int_0^\infty \frac{x^2\log x}{(x^2+1)^2}dx = \frac{\pi}{4}, \quad \int_0^\infty \frac{x^2dx}{(x^2+1)^2} = \frac{\pi}{4}$$