18.9.2017

Q7 §§7.7 Compute $\int_0^\infty \frac{\log(px)\,dx}{x^2+q^2}$ using the method of contour integration. ($p>0$ $q>0$)

We take the principal value for the logarithm function and integrate $\frac{\text{Log}(pz)}{z^2+q^2}$ around closed contour Γ of Fig 1. Limits $R\to\infty$, $\epsilon\to 0$ are to be taken at the end. In this limit we will have

$$\lim \int_{BCA} \frac{\text{Log}(pz)}{z^2+q^2}\,dz \to 0$$

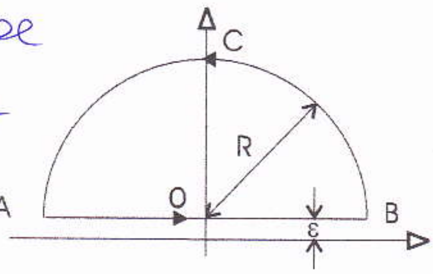

Fig. 1 Contour Γ

Therefore,

$$\oint_\Gamma \frac{\log(zp)}{z^2+q^2}\,dz = \int_{AO} \frac{\text{Log}(pz)\,dz}{z^2+q^2} + \int_{OB} \frac{\text{Log}(pz)}{(z^2+q^2)}\,dz$$

$$= \int_{OB} \frac{\text{Log}(pz)\,dz}{(z^2+q^2)} - \int_{OA} \frac{\text{Log}(pz)}{(z^2+q^2)}\,dz$$

Along OB

$z = r$ $\quad \text{Log}(pz) = \text{Log}(pr)$

$dz = dr$

Along OA

$z = re^{i\pi}$ $\quad 0<r<R$

$dz = dr\,e^{i\pi}$

$\text{Log}(pz) = \text{Log}(pr) + i\pi$

$$\therefore \oint_\Gamma \frac{\text{Log}(zp)}{(z^2+q^2)}\,dz = 2\int_0^\infty \frac{\text{Log}(pr)}{r^2+q^2}\,dr + i\pi\int_0^\infty \frac{dz}{z^2+q^2}$$

p2/Q7/§§7.7

The integral on the left hand side is computed using residue theorem. There are two poles at $z = \pm iq$ only one of these, iq, is enclosed inside the contour Γ. Therefore

$$\oint_{\Gamma} \frac{\mathrm{Log}(pz)}{z^2+q^2}\,dz = 2\pi i \times \mathrm{Res}\left\{\frac{\mathrm{Log}(pz)}{z^2+q^2}\right\}_{z=iq}$$

$$= 2\pi i \frac{\left(\mathrm{Log}\, pq + i\pi/2\right)}{2iq}$$

$$= \frac{\pi}{q}\left(\mathrm{Log}(pq) + i\pi^2\right)$$

$$\therefore \int_0^{\infty} \frac{\mathrm{Log}(pz)}{z^2+q^2}\,dz = \frac{\pi}{q}\left(\mathrm{Log}(pq) + \frac{i\pi^2}{2}\right)$$

$$2\int_0^{\infty} \frac{\log(px)}{x^2+q^2} + i\pi\int_0^{\infty} \frac{dx}{x^2+q^2} = \frac{\pi}{q}\left(\log(pq) + \frac{i\pi^2}{2}\right)$$

Equating real parts we get

$$2\int_0^{\infty} \frac{\log(px)}{x^2+q^2}\,dx = \frac{\pi}{(2q)}\log(pq)$$

The imaginary parts give the value $\pi/2q$ for $\int_0^{\infty} \frac{dx}{x^2+q^2}$.