Q4,5,6 §§7.7 The integrals appearing in these problems are best computed by considering $\int_0^\infty \frac{\log^n x}{(x^2+1)}\,dx \equiv I_n$. Use of contour Γ gives recurrence relation for I_n which can be used to get answers for higher powers of n in terms of I_0, I_1, I_2 (I_1, I_2 are known from Q1, Q2)

We take the principal branch for the logarithm function and integrate

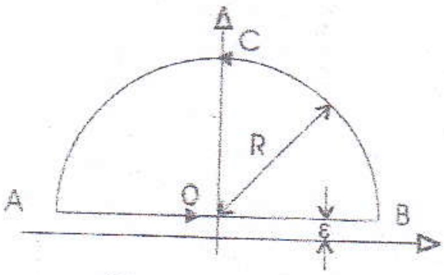

Fig. 1 Contour Γ

$$J_n = \oint_\Gamma \frac{(\mathrm{Log}\, z)^n}{z^2+1}$$

around the contour Γ of fig 1 $\int_{BCA} \frac{(\mathrm{Log}\, z)^n}{z^2+1}\,dz \to 0$ as $R\to\infty$

$$J_n = \int_{AO} \frac{(\mathrm{Log}\, z)^n}{(z^2+1)}\,dz + \int_{OB} \frac{(\mathrm{Log}\, z)^n}{(z^2+1)}\,dz$$

$$= \int_{OB} \frac{(\mathrm{Log}\, z)^n dz}{(z^2+1)} - \int_{OA} \frac{(\mathrm{Log}\, z)^n dz}{(z^2+1)}$$

We set up the two line integrals using

OA: $z = -r$, $dz = dr$, $\mathrm{Log}\, z = \ln r + i\pi$

OB: $z = r$, $dz = dr$, $\mathrm{Log}\, z = \ln r$

valid in the limit $\epsilon\to 0$. Therefore in limit $R\to\infty$, $\epsilon\to 0$ we would get

$$J_n = \int_0^\infty \frac{(\ln r)^n}{r^2+1} dr + \int_0^\infty \frac{(\ln + i\pi)^n dr}{(r^2+1)}$$

$$= 2I_n + {}^nC_1 (i\pi) I_{n-1} + {}^nC_2 (i\pi)^2 I_{n-2} + (i\pi)^n I_0. \qquad I_0 = \int_0^\infty \frac{dr}{r^2+1} = \pi/2$$

Next we compute J_n using Cauchy residue theorem. There is only one simple pole enclosed by contour Γ.

$$J_n = (2\pi i) \operatorname{Res}\left\{\frac{(\operatorname{Log} z)^n}{(z^2+1)}(z-i)\right\}_{z=i} \qquad \operatorname{Log} z\Big|_{z=i} = \frac{i\pi}{2}$$

$$= 2\pi i \left(\frac{i\pi}{2}\right)^n \times \frac{1}{2i}$$

$$= \frac{\pi^{n+1}}{2^n} i^n.$$

Therefore we get

$$2I_n + {}^nC_1 (i\pi) I_{n-1} + {}^nC_2 (i\pi)^2 I_{n-2} + \dots + (i\pi)^n I_0 = \frac{\pi^{n+1} i^n}{2^n}.$$

~~Using $n=2$ we get $2I_2 + 2\pi i I_1 \quad \pi^2 I_0$~~

For $n=0,1,2$, we recover results obtained in Q[1], Q[2]

$$2I_0 = \pi, \quad 2I_1 + (i\pi) I_0 = \frac{\pi^2 i}{2}, \quad 2I_2 + 2(i\pi)I_1 + (i\pi)^2 I_0 = \frac{\pi^3 i^2}{2^2}$$

giving $I_0 = \pi/2$, $I_1 = 0$, $I_2 = \frac{1}{2}\left(\frac{\pi^3}{2} - \frac{\pi^3}{4}\right) = \frac{\pi^3}{8}$

For Q[4]-[6] we successive take $n = 4, 6, 8$

Setting $n=4$ in the recurrence relation for I_n

$$2I_4 + {}^4C_1(i\pi)I_3 + {}^4C_2(-\pi)^2 I_2 + {}^4C_3(i\pi)^3 I_1 + \pi^4 I_0 = \frac{\pi^5}{24}$$

Considering the real part of this equation we get

$$2I_4 + \frac{4(4-1)}{2}(-\pi^2)\frac{\pi^3}{8} + \pi^4\left(\frac{\pi}{2}\right) = \frac{\pi^5}{16}$$

$$2I_4 = \pi^5\left(\frac{3}{4} - \frac{1}{2} + \frac{1}{16}\right) \Rightarrow I_4 = \frac{5}{32}\pi^5$$

Next set $n=6$ in the recurrence relation for I_n, take real part

$$2I_6 + {}^6C_2(-\pi)^2 I_4 + {}^6C_4(i\pi)^4 I_2 + (i\pi)^6 I_0 = \frac{\pi^7 (i)^6}{26}$$

$$2I_6 + \frac{6(6-1)}{2}(-\pi^2)\frac{5\pi^5}{32} + \frac{6(6-1)(6-2)(6-3)}{4.3.2.1}(i\pi)^4\frac{\pi^3}{8} + (-\pi^6)\frac{\pi}{2} = -\frac{\pi^7}{26}.$$

$$2I_6 - \left(15.\frac{5\pi^7}{32}\right) + \left(\frac{\cancel{6}^{3}.5.\cancel{4}.\cancel{3}}{\cancel{4}.\cancel{3}.\cancel{2}.1}\right)\frac{\pi^7}{8} - \frac{\pi^7}{2} = -\frac{\pi^7}{64}$$

$$\therefore I_6 = \frac{\pi^7}{2}\left(\frac{75}{32} - \frac{15}{8} + \frac{1}{2} - \frac{1}{64}\right) = \frac{\pi^7}{128}(150-120+32-1)$$

$$= \frac{61\pi^7}{128}$$

We leave for the reader to verify that $I_8 = \frac{1385}{512}\pi^9$.