11-9-2017

P1/Q3/8877

Q3 / 8877 Integrate $\int_0^\infty \frac{\log^2 x}{(x^2+1)}\,dx$ using the method of contour integration

Use contour Γ of Fig 1 and integrate $f(z) = \frac{(\log z)^2}{(z^2+1)}$ around the contour. The branch cut for $\log z$ can be taken along any line $\theta = \theta_0$ outside the upper half plane. So we take the principal value

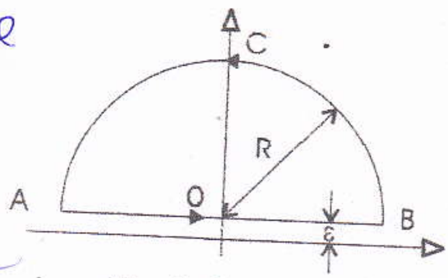

Fig. 1 Contour Γ

$\mathrm{Log}\, z = \ln r + i\theta \qquad -\pi < \theta < \pi$

$\int_{BCA} \frac{f(z)\,dz}{(z^2+1)}$ will go to zero in the limit $R \to \infty$

$\therefore \oint_\Gamma f(z)\,dz = \int_{AO} f(z)\,dz + \int_{OB} f(z)\,dz$

Now we have $= -\int_{OA} f(z)\,dz + \int_{OB} f(z)\,dz \qquad \text{--- (1)}$

<u>OA:</u> $z = -r \qquad 0 < r < \infty$

$dz = -dr$

$f(z) = \frac{(\ln r + i\pi)^2}{(r^2+1)}$

<u>For OB:</u> $z = r$

$dz = dr$

$f(z) = \frac{\ln r}{(r^2+1)}$

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Substituting in (1), in the limit $R \to \infty, \epsilon \to 0$,

$$\oint_{\Gamma} f(z)\,dz = \int_0^{\infty} \frac{(\ln r + i\pi)^2}{(r^2+1)} dr + \int_0^{\infty} \frac{\ln r^2}{(r^2+1)} dr$$

$$= 2\int_0^{\infty} \frac{(\ln r)^2}{(r^2+1)} dr + (2i\pi)\int_0^{\infty} \frac{\ln r}{(r^2+1)} dr + (i\pi)^2 \int_0^{\infty} \frac{dr}{r^2+1} \quad \text{---(2)}$$

The left hand side is computed using the residue theorem

$$\oint_{\Gamma} f(z)\,dz = 2\pi i \times \operatorname{Res}\left\{\frac{(\operatorname{Log} z)^2}{(z^2+1)}\right\}_{z=i}$$

$$= 2\pi i \,\frac{(i\pi/2)^2}{2i}$$

$$= -\frac{\pi^3}{4}$$

$$\operatorname{Res}\left\{\frac{(\log z)^2}{(z^2+1)}\right\}_{z=i} = \lim_{z\to i} (z-i)\frac{(\operatorname{Log} z)^2}{z^2+1} = \lim_{z \to i} \frac{(\operatorname{Log} z)^2}{z+i} = \frac{(i\pi/2)^2}{2i}$$

Substituting in (1)

$$2\int_0^{\infty} \frac{(\ln x)^2}{x^2+1} dx + (2i\pi)\int_0^{\infty} \frac{\ln x}{(x^2+1)} dx - \pi^2 \int_0^{\infty} \frac{dx}{(x^2+1)} = -\frac{\pi^2}{4}$$

Equating real and imaginary parts

$$\int_0^{\infty} \frac{(\log x)^2}{(x^2+1)} = \pi^3 \qquad \int_0^{\infty} \frac{\log x}{(x^2+1)} dx = 0$$

Where we have used $\int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$.