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§§7.7 Q2 Compute the integral $\int_0^\infty \frac{\log x}{x^3+1}\,dx$ by the method of contour integration

We set up a contour integral of $f(z) = \frac{\log z}{z^3+1}$ around the closed contour OBCDO of fig 1. where $\angle DOB = \frac{2\pi}{3}$. (What happens if we try to use contour AOBCA?)

The branch cut for $\log z$ can be taken along any ray θ_0 with $\frac{2\pi}{3} < \theta_0 < 2\pi$. We take principal value

$$\mathrm{Log}\, z = \ln r + i\theta, \quad -\pi < \theta < \pi.$$

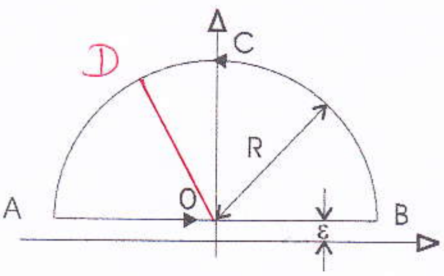

Fig. 1 Contour Γ

$$\oint_{OBCDO} f(z)dz = \int_{OB} f(z)dz + \int_{BCD} f(z)dz + \int_{DO} f(z)dz$$

$$= \int_{OB} f(z)dz - \int_{OD} f(z)dz \qquad \int_{BCD} \to 0 \text{ in limit } R\to\infty.$$

To set up the two integrals along OB and OD we use

$\underline{OB}$ $\quad z = r \quad 0 < r < R$ $\qquad$ $\underline{OD}$ $\quad z = e^{2\pi i/3} r$

$dz = dr$ $\qquad$ $dz = e^{2\pi i/3} dr$

$\mathrm{Log} z = \ln r$ $\qquad$ $\mathrm{Log} z = \ln r + \frac{2\pi i}{3}$

$$\int_{OB} f(z)dz = \int_0^R \frac{\ln r}{r^3+1}\,dr$$

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$$\int_{OD} f(z)dz = \int_0^R \frac{(\ln r + \frac{2\pi i}{3})}{r^3+1} e^{2\pi i/3} dr$$

$$= e^{2\pi i/3} \int_0^R \frac{\ln r}{r^3+1} dr - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^R \frac{dr}{r^3+1}$$

Taking limit $R \to \infty$, $\epsilon \to 0$, and using notation

$$I_1 = \int_0^\infty \frac{\ln r}{r^3+1} dr \qquad I_2 = \int_0^\infty \frac{dr}{r^3+1}$$

We get

$$I_1 (1 - e^{2\pi i/3}) - \frac{2\pi i}{3} e^{2\pi i/3} I_2$$

$$= \oint_{OBCDO} \frac{\operatorname{Log} z \, dz}{z^3+1} = 2\pi i \times \text{sum of residues of } f(z) \quad \text{---(1)}$$

Only one pole at $z = e^{i\pi/3}$ is enclosed by the contour

$$\operatorname{Res}\{f(z)\}_{z=e^{i\pi/3}} = \lim_{z \to e^{i\pi/3}} \frac{(z - e^{i\pi/3}) \operatorname{Log} z}{z^3+1}$$

$$= \lim_{z \to e^{i\pi/3}} \frac{(z - e^{i\pi/3})}{z^3+1} \cdot \lim_{z \to e^{i\pi/3}} \operatorname{Log} z$$

$$= \left. \frac{1}{3z^2} \right|_{z=e^{i\pi/3}} \times i\pi/3$$

$$= \left(\frac{i\pi}{3}\right) \times \frac{1}{3} e^{-2\pi i/3} = \frac{i\pi}{9} e^{-2\pi i/3} \quad \text{---(2)}$$

Substituting (2) in (1) we get

$$I_1(1-e^{2\pi i/3}) + \frac{2\pi i}{3}e^{2\pi i/3}I_2$$
$$= 2\pi i \times \frac{i\pi}{9}\, e^{-2\pi i/3}$$

$$I_1\left(\frac{3}{2}-\frac{i\sqrt{3}}{2}\right) - \frac{2\pi i}{3}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)I_2$$
$$= -\frac{2\pi^2}{9}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$$

$$\frac{3}{2}I_1 - \frac{i\sqrt{3}}{2}I_1 + \frac{\pi i}{3}I_2 + \frac{\pi}{\sqrt{3}}I_2 = \frac{2\pi^2}{9}\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$$

Equating real and imaginary parts we get

$$\frac{3}{2}I + \frac{\pi}{\sqrt{3}}I_2 = \frac{\pi^2}{9}$$
$$-\frac{\sqrt{3}}{2}I_1 + \frac{\pi}{3}I_2 = \frac{\pi^2\sqrt{3}}{9}$$

Solving for I_1 and I_2 gives

$$I_1 = -\frac{2\pi^2}{27}, \quad I_2 = \frac{2\pi}{3\sqrt{3}}$$

$$\therefore \int_0^\infty \frac{\ln x}{x^3+1} = -\frac{2\pi^2}{27}.$$