Q12
§§7.4

Compute the integral

$$I=\int_0^\infty \frac{x\sin 2ax\cos^2 bx}{(\beta^2+x^2)}\,dx$$

The strategy for this integral is same as used for Q11. Note that

$x\sin 2ax\cos^2 bx$

$=\frac{1}{2}\sin 2ax(1+\cos 2bx)$

$=\frac{1}{2}\left(\sin 2ax+\frac{1}{2}\sin(2(a+b)x)+\frac{1}{2}\sin 2(a-b)x\right)$

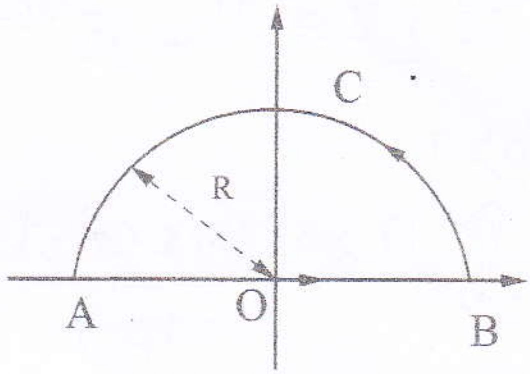

Fig. 1 Semi-circular contour

$=\frac{1}{4}\left(2\sin 2ax+\sin(2(a+b)x)+\frac{1}{2}\sin(2(a-b)x)\right)$

At first we shall compute $\int_0^\infty \frac{x\sin px}{(x^2+\beta^2)}\,dx$, $p>0, \beta>0$.

$$\int_0^\infty \frac{x\sin px}{(x^2+\beta^2)}\,dx=\frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{x\,e^{ipx}\,dx}{(x^2+\beta^2)},\quad p>0$$

$$=\frac{1}{2}\,\mathrm{Im}\lim_{R\to\infty}\oint_{AOBCA}\frac{z\,e^{ipz}}{(z^2+\beta^2)}\,dz$$

$$=\frac{1}{2}\,\mathrm{Im}.\ 2\pi i\ \mathrm{Res}\,\frac{z\,e^{ipz}}{(z^2+\beta^2)}\Big|_{z=i\beta}$$

$$=\frac{1}{2}\,\mathrm{Im}\,\frac{2\pi i\,(i\beta)\,e^{-\beta p}}{2i\beta}=\frac{\pi}{2}e^{-\beta p}.$$

Therefore the required integral is given by

$\underline{a>b}$

$$\int_0^\infty \frac{x \sin 2ax \cos^2 bx}{(x^2+\beta^2)} dx$$

$$= \frac{\pi}{8}\left[2e^{-a\beta} + e^{-2(a+b)\beta} + e^{-2(a-b)\beta}\right]$$

$b>a$

$$\int_0^\infty \frac{x \sin 2ax \cos^2 bx}{(x^2+\beta^2)} dx$$

$$= \frac{\pi}{8}\left[2e^{-a\beta} + e^{-2(a+b)\beta} + e^{-2\beta(b-a)\beta}\right]$$

for $a=b$ we need to start from the begining

$$x \sin 2ax \cos^2(xa)$$

$$= \frac{1}{2} \sin 2ax (1+\cos 2ax)$$

$$= \frac{1}{2}(\sin 2ax + \tfrac{1}{2}\sin 4ax)$$

$$= \frac{1}{4}(2\sin 2ax + \sin 4ax)$$

and we would get

$$\int_0^\infty \frac{x \sin 2ax \cos^2 bx}{(x^2+\beta^2)} dx = \frac{\pi}{8}\left(e^{-2\beta a} + e^{-4\beta a}\right)$$