§§7.4 Q9 Integrate $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2+c^2}\,dx$, $a, b, c > 0$ using the method of contour integration.

The given integral is seen to be real part of contour integral

$$J = \oint_{AOBCA} \frac{e^{iaz}\,dz}{(z^2+b^2)^2+c^2}$$

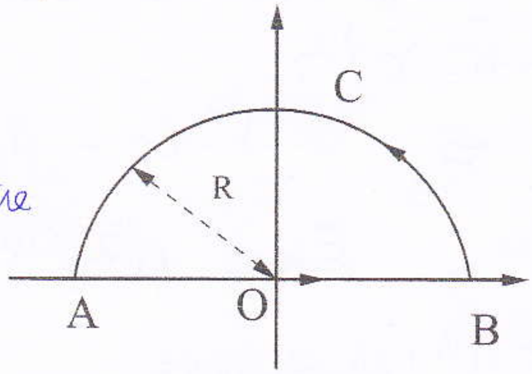


Fig. 1 Semi-circular contour

The integrand has poles where the denominator vanishes

$$(z^2+b^2)^2 + c^2 = 0 \qquad \text{——(1)}$$

$$(z^2+b^2) = \pm ic$$

$$z^2 = -b^2 \pm ic$$

$$z = (-b^2 \pm ic)^{1/2}$$

We write

$$-b^2 \pm ic = \rho_1 e^{-i\phi_1 + i\pi} \qquad \text{—(2)}$$

where $\rho_1 = \sqrt{b^4+c^2}$

$$\phi_1 = \tan^{-1}(c/b^2)$$

Remarks:

The computation does not require any new technique. We only need to keep track of residues at poles enclosed and simplify the final result to obtain the desired form.

Since c and b^2 are real positive, ϕ_1 is between 0 and $\pi/2$

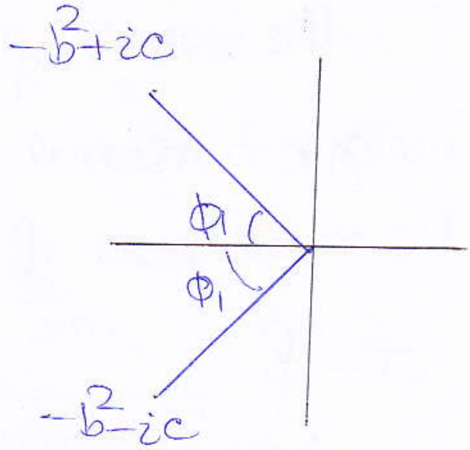
∴ ~~The~~ Two of the four roots are

$$\left.\begin{aligned} \xi_1 &= (-b^2+ic)^{1/2} = \rho^{1/2} e^{-\frac{i\phi}{2}} e^{i\pi/2} \\ \text{and}\quad \xi_2 &= (-b^2+ic)^{1/2} = -\rho^{1/2} e^{-i\phi/2} e^{i\pi/2} \end{aligned}\right\} \quad (3)$$

ξ_1 is seen to be in the first quadrant and ξ_2 is in the third quadrant.

In the above notation for ρ, ϕ we have

$$(-b^2-ic) = \rho_1 e^{i\pi} e^{i\phi}$$



Therefore the other two roots of the eqn (1) are values of $(-b^2-ic)^{1/2}$.

$$\xi_3 \qquad -b^2-ic = \rho e^{i\phi} e^{i\pi}$$

$$\xi_3 = \rho^{1/2} e^{i\phi/2} e^{i\pi/2} \qquad \xi_4 = -\rho^{1/2} e^{i\phi/2} e^{i\pi/2} \quad \text{—(4)}$$

Eqn (2) shows

$$\rho^2 = \sqrt{b^4+c^2}$$

Also $\tan\phi = \dfrac{b^2}{\sqrt{b^4+c^2}}$ gives

$$\cos^2\phi/2 = \tfrac{1}{2}(1+\cos\phi) = \tfrac{1}{2}(\sqrt{b^4+c^2}+b^2)/\rho$$

$$\equiv A^2/\rho$$

$$\sin^2\phi/2 = \tfrac{1}{2}(1-\cos 2\phi) = \tfrac{1}{2}(\sqrt{b^4+c^2}-b^2)/\rho$$

$$\equiv B^2/\rho$$

$$\cos(\phi/2) = A/\rho^{1/2} \qquad \sin(\phi/2) = B/\rho^{1/2} \qquad (6)$$

Where A and B are given by

$$2A^2 = (\sqrt{b^4+c^2}+b^2), \quad 2B^2 = (\sqrt{b^4+c^2}-b^2)$$

The four roots ξ_1, ξ_2, ξ_3 and ξ_4 can not be written as

$$\xi_1 = \rho^{1/2} e^{i\pi/2} e^{-i\phi/2}$$

$$= i\rho^{1/2}\left(\cos\phi/2 - i\sin(\phi/2)\right)$$

$$\xi_1 = iA + B$$

Similarly $\xi_2 = (iA - B)$, $\xi_3 = (-iA - B)$, $\xi_4 = (-iA + B)$

Note that ξ_1 and ξ_2 are in upper half plane $(A > 0)$

$$\text{Residue } \left.\frac{e^{iaz}}{(z^2+b^2)^2+c^2}\right|_{z=\xi_1}$$

$$= \lim_{z\to\xi_1} \frac{e^{iaz}(z-\xi_1)}{(z^2+b^2)^2+c^2} = \lim e^{iaz} \cdot \lim \frac{z-\xi_1}{(z^2+b^2)^2+c^2}$$

$$= e^{ia\xi_1} \times \left.\frac{1}{2z \cdot 2(z^2+b^2)}\right|_{\xi_1}$$

$$= \frac{e^{-Aa+iaB}}{4(B+iA)(ic)}$$

$\because z^2+b^2 = ic$

Similarly the residue at $z = \xi_2$ is given by

$$\text{Res } \left.\frac{e^{iaz}}{(z^2+b^2)^2+c^2}\right|_{z=\xi_2} = \frac{e^{-aA-iaB}}{4(-B+iA)(-ic)}$$

Therefore the contour integral J is given by

$$J = \frac{2\pi i\, e^{-aA}}{4c}\left(\frac{\cos aB + i\sin aB}{(-A+iB)} + \frac{(\cos(aB) - i\sin(aB))}{A+iB}\right)$$

$$= \frac{\pi i}{2c}\frac{e^{-Aa}}{(-A^2-B^2)}\Big((A+iB)(\cos(Ba) + i\sin(aB))$$

$$+ (-A+iB)(\cos(aB) - i\sin(aB))\Big)$$

$$= \frac{-\pi i}{2c}\frac{e^{-aA}}{(A^2+B^2)}\, 2i\,(A\sin(aB) + B\cos(aB))$$

$$= \frac{\pi}{c}\frac{e^{-aA}}{(A^2+B^2)}(B\cos(aB) + A\sin(aB))$$

The required integral is

$$= \frac{1}{2}\operatorname{Re} J$$

$$= \left(\frac{\pi}{2c}\right)\frac{e^{-aA}}{\sqrt{b^4+c^2}}(B\cos(aB) + A\sin(aB))$$

20-5-2017