§§ 7.4 Q 7

Compute the integral

$$\int_{-\infty}^{\infty} \frac{x\cos\alpha x\,dx}{x^2-2x+10} \qquad \alpha = \text{real}$$

by the method of contour integration.

The given integral is equal to the real part of $\int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^2-2x+10}\,dx$

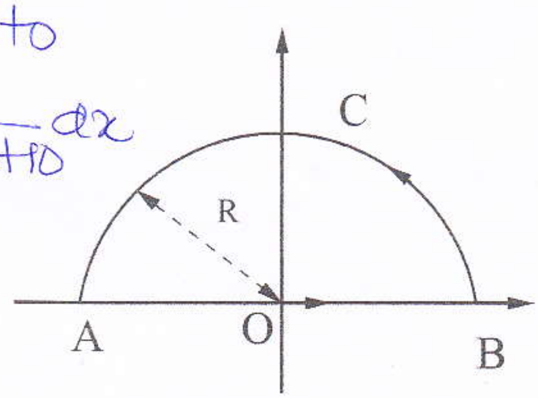

Fig. 1 Semi-circular contour

$$\int_{-\infty}^{\infty} \frac{x\cos\alpha x}{x^2-2x+10}\,dx$$

$$= \mathrm{Re}\int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^2-2x+10}$$

$$= \mathrm{Re}\oint_{AOBCA} \frac{z e^{i\alpha z}}{z^2-2z+10}\,dz$$

∵ Since $\alpha > 0$ the contour integral along BCA $\to 0$ as $R \to \infty$

$$= 2\pi i \sum_{\substack{\text{residues}\\ \text{in upper half}\\ \text{plane}}} \frac{z e^{i\alpha z}}{z^2-2z+10}$$

$z^2-2z+10=0$

$\Rightarrow z = \frac{2 \pm i\sqrt{36}}{2}$

$= 1 \pm i3$

$= \xi, \xi^*$

$\xi - \xi^* = 6i$

$$= 2\pi i \operatorname*{Res}_{z=1+i\sqrt{3}} \left(\frac{z e^{i\alpha z}}{z^2-2z+10} \right)$$

$$= 2\pi i \left. \frac{(z-\xi)\, z\, e^{i\alpha\xi}}{(z-\xi)(z-\xi^*)} \right|_{z=\xi}$$

$$= 2\pi i\; \xi e^{i\alpha\xi}/(\xi-\xi^*) = \frac{\pi}{3} e^{i\alpha-3\alpha}(1-3i)$$

∴ Required integral is given by

$$\int_{-\infty}^{\infty} \frac{x\cos\alpha x}{x^2-2x+10}\,dx$$

$$= \mathrm{Re} \oint_{AOBCA} \frac{z e^{i\alpha z}}{z^2-2z+10}\,dz$$

$$= \frac{\pi}{3} e^{-3\alpha}\, \mathrm{Re}\, e^{i\alpha}(1+3i)$$

$$= \frac{\pi}{3} e^{-3\alpha}(\cos\alpha - 3\sin\alpha)$$