§§ 7.4
Q 6

Compute $\displaystyle\int_0^\infty \frac{x \sin px}{(x^4+1)}\,dx$ by the method of contour integration.

Since the integrand is an even function of x, we can change the integration limits to $(-\infty, \infty)$ and write

$$\int_0^\infty \frac{x\sin px}{(x^4+1)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin px}{(x^4+1)}\,dx$$

$$= \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{x\, e^{ipx}}{(x^4+1)}\,dx$$

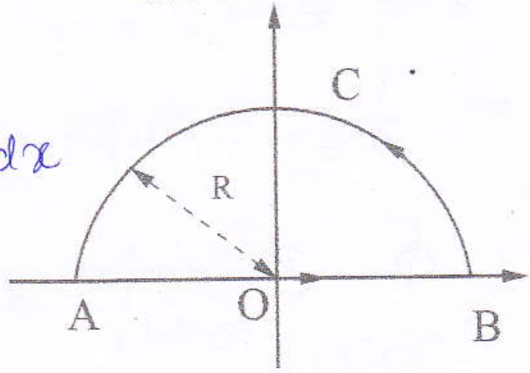

Fig. 1 Semi-circular contour

The last integral equals to a contour integral along AOBCA in the limit $R\to\infty$

$$\int_0^\infty \frac{x\sin px}{x^4+1}\,dx = \frac{1}{2}\,\mathrm{Im}\lim_{R\to\infty}\oint_{AOBCA} \frac{z\, e^{ipz}}{(z^4+1)}\,dz$$

The integrand has four poles at $\xi_k = e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{-i\pi/4}$, $e^{-3i\pi/4}$. Of these only the first two are enclosed by the contour AOBCA. Computing residues at these two points we get

$$\mathrm{Res}\, \frac{z e^{ipz}}{(z^4+1)}\Big|_{z=\xi}$$

$\xi_1 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

$\xi_2 = e^{3i\pi/4} = \frac{-1+i}{\sqrt{2}}$

$\xi_1^2 = i$, $\xi_2^2 = -i$

$\xi^4 = -1$

$$= \lim_{z\to\xi} (z-\xi) \frac{z e^{ipz}}{(z^4+1)}$$

$$= \lim_{z\to\xi} \left(\frac{z-\xi}{z^4+1}\right) \lim_{z\to\xi} \left(z e^{ipz}\right)$$

$$= \frac{1}{4\xi^3} \xi e^{ip\xi} = \frac{1}{4\xi^2} e^{ip\xi}$$

$$\therefore \oint_{AOBCA} \frac{z e^{ipz}}{(z^4+1)} dz = 2$$

$$= 2\pi i \left(\frac{1}{4\xi_1^2} e^{ip\xi_1} + \frac{1}{4\xi_2^2} e^{ip\xi_2} \right)$$

$$= \frac{2\pi i}{4} \left(-i\, e^{ip(1+i)/\sqrt{2}} + i\, e^{ip(-1+i)/\sqrt{2}} \right)$$

$$= \frac{\pi}{2} e^{-p/\sqrt{2}} \left(e^{ip/\sqrt{2}} - e^{-ip/\sqrt{2}} \right)$$

$$= \frac{2\pi i}{2} e^{-p/\sqrt{2}} \sin(p/\sqrt{2})$$

$$\therefore \int_0^\infty \frac{x \sin px}{(x^4+1)} dx = \frac{1}{2} \mathrm{Im} \oint \frac{z^2 e^{ipz}}{(z^4+1)} dz$$

$$= \frac{1}{2} \pi e^{-p/\sqrt{2}} \sin(p/\sqrt{2})$$