$\boxed{\begin{array}{c}\S\S 7.4 \\ Q4\end{array}}$ Compute the integral $\int_0^\infty \frac{x^3 \sin px\, dx}{(x^2+4)(x^2+9)}$ by the method of contour integration.

Proceeding as in Q[1] we have

$$\int_0^\infty \frac{x^3 \sin px\, dx}{(x^2+4)(x^2+9)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^3 \sin px\, dx}{(x^2+4)(x^2+9)}$$

$$= \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{x^3 e^{ipx}\, dx}{(x^2+4)(x^2+9)}$$

$$= \frac{1}{2}\,\mathrm{Im}\oint_{AOBCA} \frac{z^3 e^{ipz}\, dz}{(z^2+4)(z^2+9)}$$

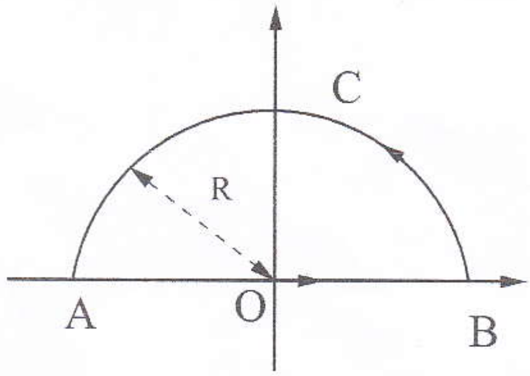

Fig. 1 Semi-circular contour

Now $\oint_{AOBCA} \frac{z^3 e^{ipz}\, dz}{(z^2+4)(z^2+9)}$

$$= 2\pi i \times \sum_{\substack{\text{residues} \\ \text{at } z=2i,3i}} \frac{z^3 e^{ipz}}{(z^2+4)(z^2+9)}$$

Compute the required residues

$$\operatorname*{Res}_{z=2i} \frac{z^3 e^{ipz}}{(z^2+4)(z^2+9)} = \left.\frac{(z-2i)\, z^3 e^{ipz}}{(z^2+4)(z^2+9)}\right|_{z\to 2i}$$

$$= \left.\frac{z^3 e^{ipz}}{(z+2i)(z^2+9)}\right|_{z\to 2i} = \frac{(8i^3)\, e^{-2p}}{(4i)\times 5} = -\frac{2}{5} e^{-2p}$$

$$\operatorname*{Res}_{z=3i} \left.\frac{z^3 e^{ipz}}{(z^2+4)(z^2+9)}\right| = \left.\frac{(z-3i)(z^3 e^{ipz})}{(z^2+4)(z^2+9)}\right|_{z\to 3i}$$

$$= \left.\frac{z^3 e^{ipz}}{(z^2+4)(z+3i)}\right|_{z\to 3i} = \frac{(27 i^3)\, e^{-3p}}{-5\times 6i} = \frac{9}{10} e^{-3p}$$

Therefore

$$\oint_{AOBCA} \frac{z^3 \sin pz \, dz}{(z^2+4)(z^2+9)} = \left(\frac{9}{10}e^{-3p} - \frac{2}{5}e^{-p}\right)$$

$\therefore$ Required Integral

$$= \operatorname{Im} 2\pi i \times \frac{1}{2} \times \left(\frac{9}{10}e^{-3p} - \frac{2}{5}e^{-p}\right)\Big|_{p=1}$$

$$= \operatorname{Im} \pi i \left(\frac{9}{10}\right)\left(e^{-3p} - 4e^{-p}\right)\Big|_{p=1}$$

$$= \frac{9\pi}{10e^3}\left(9 - 4e\right)$$