§§7.4 Q3

Compute the following integral using the method of contour integration

$$\int_0^\infty \frac{\cos px \, dx}{(x^2+1)(x^2+4)}$$

Using I to denote the integral we write it as

$$I = \int_0^\infty \frac{\cos px \, dx}{(x^2+1)(x^2+4)}$$

$$= \frac{1}{2} \mathrm{Re} \int_{-\infty}^\infty \frac{e^{ipx} \, dx}{(x^2+1)(x^2+4)}$$

$$= \frac{1}{2} \mathrm{Re} \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ipx} \, dx}{(x^2+1)(x^2+4)}$$

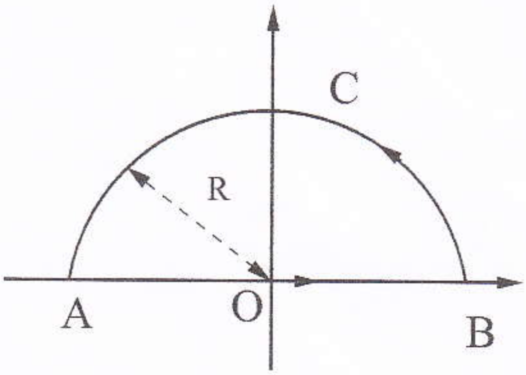

Fig. Semi-circular contour

$$= \frac{1}{2} \mathrm{Re} \lim_{R\to\infty} \int_{AOB} \frac{e^{ipz} \, dx}{(z^2+1)(z^2+4)}$$

$\because$ along AOB $z = x$, $dz = dx$

$$= \frac{1}{2} \mathrm{Re} \lim_{R\to\infty} \oint_{AOBCA} \frac{e^{ipz} \, dz}{(z^2+1)(z^2+4)}$$

$\because \int_{BCA} \frac{e^{ipz} dz}{(\,)(\,)} \to 0$ as $R \to \infty$

To evaluate the integral $\oint_{AOBCA}$, we need to compute the residues at the poles enclosed inside the contour

$$\mathrm{Res} \left. \frac{e^{ipz}}{(z^2+1)(z^2+4)} \right|_{z=i} = \lim_{z\to i} \frac{(z-i)\, e^{ipz}}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to i} \frac{e^{ipz}}{(z+i)(z^2+4)} = \frac{e^{-p}}{2i \times 3}$$

$$\operatorname*{Res}_{z=2i} \frac{e^{ipz}}{(z^2+1)(z^2+4)} = \left.\frac{(z-2i)\,e^{ipz}}{(z^2+1)(z^2+4)}\right|_{z=2i}$$

$$= \frac{e^{-2p}}{(-3)\times(4i)} = \frac{i e^{-2p}}{12}$$

$$\therefore \oint_{AOBCA} \frac{e^{ipz}\,dz}{(z^2+1)(z^2+4)} = 2\pi i \times \left(\frac{e^{-p}}{6i} + \frac{i e^{-2p}}{12}\right)$$

$$= \frac{\pi}{6}\left(2e^{-p} - e^{-2p}\right)$$

$$\therefore I = \frac{\pi}{12}\left(2e^{-p} - e^{-2p}\right)$$

$\boxed{\text{May 10, 2017}}$