§§7.4 Q1 Compute $\int_0^\infty \frac{x \sin px}{(x^2+1)}\,dx$ using the method of contour integration

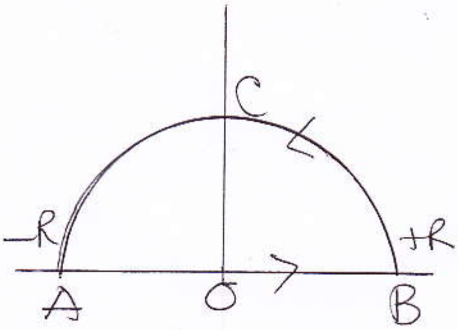
Following the method of Problem §1.1.2 in part I of the book, the integral can be seen to be related to the imaginary part of the integral $\oint_{AOBCA} \frac{z e^{ipz}}{(z^2+1)}\,dz$.

To see this we transform the given integral as follows.

$$\int_0^\infty \frac{x \sin px}{(x^2+1)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin px}{(x^2+1)}\,dx$$

$$= \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{x e^{ipx}}{x^2+1}\,dx$$

$$= \frac{1}{2}\,\mathrm{Im}\lim_{R\to\infty}\int_{-R}^{R} \frac{z e^{ipz}}{(z^2+1)}\,dz$$



In the limit $R \to \infty$, a semicircular contour can be added without changing the value of the integral

$$\therefore \int_0^\infty \frac{x \sin px}{(x^2+1)}\,dx = \frac{1}{2}\,\mathrm{Im}\lim_{R\to\infty}\oint_{AOBCA} \frac{z e^{ipz}}{(z^2+1)}\,dz$$

The integrand on the right-hand side has only one singular point inside the contour AOBCA (at $z=i$)

$$\therefore \int_0^\infty \frac{x \sin px}{(x^2+1)}\,dx = \frac{1}{2}\,\mathrm{Im}\lim_{R\to\infty} \mathrm{Res}\left(\frac{z e^{ipz}}{(z^2+1)}\right)\bigg|_{z=i} \times 2\pi i$$

$$= \frac{1}{2}\,\mathrm{Im}\; \frac{(z-i) z e^{ipz}}{(z-i)(z+i)}\bigg|_{z=i} \times 2\pi i = \frac{\mathrm{Im}}{2}\left(\frac{i e^{-p}}{2i} \times 2\pi i\right) = \frac{\pi}{2} e^{-p}.$$

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