2-10-2017

Given partial differential equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

Separate the variables by taking

$$u(x,t) = y(x)T(t)$$

The partial differential equation gives

$$T(t)\frac{d^2y}{dx^2} = \frac{1}{k}y(x)\frac{dT}{dt}$$

or $$\frac{1}{y(x)}\frac{d^2y}{dx^2} = \frac{1}{k}\frac{1}{T(t)}\frac{dT}{dt}$$

Each side must be a constant, say λ. This gives

$$\frac{d^2y}{dx^2} - \lambda y(x) = 0 \quad , \quad \frac{dT}{dt} = kT(t)\lambda$$

The solutions of the above equations are

$$y(x) = \begin{cases} Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} & \lambda \neq 0 \\ Cx + D & \lambda = 0 \end{cases}$$

$$T(t) = \begin{cases} e^{\lambda kt} & \lambda \neq 0 \\ Et & \lambda = 0 \end{cases}$$

Because solution must remain bounded as $t \to \infty$, for all $-a < x < a$. Therefore $\lambda = 0$ is ruled out.

pd-que-01003-sol.pdf

We therefore write solution as.

$$u_\lambda(x,t) = \left(A e^{\sqrt{\lambda}x} + B e^{-\sqrt{\lambda}x}\right) e^{\lambda k t}.$$

Next we impose the boundary conditions at $x = \pm a$

$$\frac{\partial}{\partial x} u_\lambda(x,t) \Big|_{x=\pm a} = 0 \quad \text{for all } t$$

This gives

$$A e^{\sqrt{\lambda}a} - B e^{-\sqrt{\lambda}a} = 0$$

$$A - B = 0$$

Therefore

$$\begin{vmatrix} 1 & -1 \\ e^{\sqrt{\lambda}a} & -e^{-\sqrt{\lambda}a} \end{vmatrix} = 0 \implies e^{2\sqrt{\lambda}a} = +1$$

Hence we get $2\sqrt{\lambda}a = 2\pi i n$ or $\sqrt{\lambda} = \frac{\pi i n}{a}$

where n is any integer. Thus we get the following solutions

$$u_n(x,t) = A\left(e^{i n\pi x/a} + e^{-i n\pi x/a}\right) e^{-n^2\pi^2 k t/a^2}$$

$$= A' \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2 k t/a^2}$$

Therefore the most general solution is a superposition

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2 k t/a^2}$$

We are given that at time $t=0$

$$u(x,t) = |x|$$

This gives

$$|x| = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-n^2\pi^2 k t/a^2}$$

The coefficients A_n are given by

$$A_0 = \frac{1}{2a}\int_{-a}^{a} f(x)\,dx$$

$$A_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi x}{a}\,dx \qquad \text{where } f(x) = |x|.$$

Given function $f(x)$ is

$$f(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$$

Therefore we should split the range of integration into ~~to~~ two parts

$$\int_{-a}^{a} \rightarrow \int_{-a}^{0} + \int_{0}^{a}$$

$$A_0 = \frac{1}{2a}\int_{-a}^{a}|x|\,dx = \frac{1}{a}\int_{0}^{a} x\,dx = \frac{a}{2}$$

$$A_n = \frac{1}{a}\int_{-a}^{a}|x|\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{1}{a}\int_{-0}^{0}(-x)\cos\frac{n\pi x}{a} + \int_{0}^{a} x\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\int_{0}^{a} x\cos\left(\frac{n\pi x}{a}\right)dx \qquad \text{(Integrate by parts)}$$

$$= \frac{2}{a}\left[\underbrace{x\sin\left(\frac{n\pi x}{a}\right)\times\left(\frac{a}{n\pi}\right)\Big|_0^a}_{=0} - \left(\frac{a}{n\pi}\right)\int_0^a \sin\frac{n\pi x}{a}\,dx\right]$$

$$= \frac{2}{(n\pi)}\left(\frac{a}{n\pi}\right)\left(\cos\frac{n\pi x}{a}\Big|_0^a\right)$$

$$= \frac{2a}{(n\pi)^2}\left(\cos n\pi - 1\right).$$

$$= \frac{2a}{(n\pi)^2}\left((-1)^n - 1\right)$$

Therefore we get

$$A_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4a}{(n\pi)^2} & \text{if } n \text{ is odd} \end{cases}$$

Hence the final answer for the solution takes the form

$$u(x,t) = \frac{2}{a} - \left(\frac{4a}{\pi^2}\right) \sum_{\text{odd } n} \frac{1}{n^2} \cos\frac{n\pi x}{a}\, e^{-n^2\pi^2 kt/a^2}$$

$$= \frac{2}{a} - \left(\frac{4a}{\pi^2}\right) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos\frac{(2n+1)\pi x}{a}\, e^{-(2n+1)^2\pi^2 kt/a^2}$$