Frobenius Method of Series Solution

Case-III Some Coefficient becomes infinite

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☹ *Question:*

For differential equation

$$x\frac{d^2y(x)}{dx^2} - x\frac{dy}{dx} - y(x) = 0$$

assuming a series solution in the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c}$$

(*a*) find the indicial equation for c and values of c;

(*b*) derive the recurrence relation;

(*c*) analyse the case of series solution to which the equation belong to;

(*d*) Show that

$$y(x,c) = a_0[1 + \frac{x}{c} + \frac{x^2}{c(c+1)} + \frac{x^3}{c(c+1)(c+2)} + \cdots]$$

(*e*) Find first few terms of the two linearly independent solutions.

☺ *Solution:*

Assume

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{1}$$

so that

$$\frac{dy(x,c)}{dx} = \sum_{n=0}^{\infty} a_n (n+c) x^{n+c-1} \tag{2}$$

$$\frac{d^2y(x,c)}{dx^2} = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} \tag{3}$$

$$\tag{4}$$

Substituting in the differential equation we get

$$x\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} - x\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{5}$$

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-1} - \sum_{n=0}^{\infty} a_n(n+c)x^{n+c} - \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{6}$$

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-1} - \sum_{n=0}^{\infty} a_n(n+c+1)x^{n+c} = 0 \tag{7}$$

(a) The minimum power of x is $(c-1)$. The coefficient of x^{c-1} equated to zero gives

$$a_0 c(c-1) = 0$$

assuming $a_0 \neq 0$ we get the indicial equation

$$c(c-1) = 0 \quad \Rightarrow \quad c = 0, 1 \tag{8}$$

(b) The coefficient of x^{c+m} equated to zero gives

$$a_{m+1}(c+m+1)(c+m) - a_m(c+m+1) = 0. \tag{9}$$

or

$$a_{m+1} = \frac{a_m}{(c+m)}.. \tag{10}$$

(c) The coefficient of x^c is given by

$$(c+1)ca_1 - a_0(c+1) = 0 \quad \Rightarrow \quad a_1 = a_0/c. \tag{11}$$

Therefore a_1 becomes infinite for $c = 0$This equation belong to CASE-III of Frobenius method.

(d) Using the recurrence relation (10) we successively get

$$\begin{aligned} a_1 &= a_0/c & (12)\\ a_2 &= a_1/(c+1) = a_0/c(c+1) & (13)\\ a_3 &= a_2/(c+2) = a_0/c(c+1)(c+2) & (14)\\ a_m &= a_{m-1}/c = a_0/c(c+1)\dots(c+m-1) & (15) \end{aligned}$$

Therefore, we get

$$y(x,c) = a_0\Big\{1 + \frac{x}{c} + \frac{x^2}{c(c+1)} + \frac{x^3}{c(c+1)(c+2)} + \cdots \frac{x^m}{c(c+1)...(c+m-1)} + \cdots\Big\} \tag{16}$$

Case $c = 1$: In this we have the first solution given by

$$y = a_0 x\Big\{1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \cdots + \frac{x^m}{1.2.3.\dots m} + \cdots\Big\} \tag{17}$$

Thus first solution is

$$y_1(x) = xe^x. \tag{18}$$

Case $c = 0$: For $c = 0$,we first substitute $a_0 = kc$ in the series for $y(x,c)$ to get

$$y(x,c) = kx^c\Big\{c + x + \frac{x^2}{(c+1)} + \frac{x^3}{(c+1)(c+2)} + \cdots \frac{x^m}{(c+1)...(c+m-1)} + \cdots\Big\} \tag{19}$$

Substituting $c = 0$ we can get solutions

$$y_{1A}(x) = y(x,c)\Big|_{c=0}, \qquad y_2(x) = \frac{dy(x,c)}{dx}\Big|_{c=0} \tag{20}$$

The solution y_{1A} is seen to coincide with $y_1(x)$ of (18). The second solution is obtained from $y_2(x)$ of (20)

$$\begin{aligned} y_2(x) &= k\log x y_1 + k\Big[1 - \frac{x^2}{(c+1)^2} + x^3\Big\{-\frac{1}{(c+1)^2(c+2)} - \frac{1}{(c+1)(c+2)^2}\Big\} + \cdots\Big]_{c=0} & (21)\\ &= kxe^x(\log x) + k\Big(1 - x^2 - \frac{3x^3}{4} + \cdots\Big) & (22) \end{aligned}$$

The two linearly independent solutions are therefore given by

$$\begin{aligned} y_1(x) &= xe^x. & (23)\\ y_2(x) &= y_1(x)\log x + \Big(1 - x^2 - \frac{3x^3}{4} + \cdots\Big). & (24) \end{aligned}$$