QM-08 Lecture Notes
Using algebraic methods for eigenvalues *

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Contents

Abstract

The canonical commutation relations are used to obtain (i) energy eignevalues and eigenvectors of harmonic oscillator and (ii) the eigenvalues and eignvectors of angular momentum operators J^2 and J_z

§1 Harmonic oscillator energy eigenvalues

Operator algebra for harmonic oscillator

The classical Hamiltonian for harmonic oscillator in one dimension is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{1}$$

The corresponding operator $\hat{H}$ is obtained by replacing the position and momentum x, p by the operators $\hat{x}, \hat{p}$ satisfying canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. Note that we will not use any explicit representation of the operators. The commutation relation is sufficient to obtain all the answers.

We introduce operators $N, a, a^\dagger$ by

$$a = \frac{1}{\sqrt{2m\omega\hbar}}(\hat{p} - im\omega\hat{x}), \tag{2}$$

$$a^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(\hat{p} + im\omega\hat{x}), \tag{3}$$

$$N = a^\dagger a. \tag{4}$$

It is easy to see that these operators satisfy the following identities:

$$[a, a^\dagger] = 1, \qquad [N, a] = -a, \qquad [N, a^\dagger] = a^\dagger, \tag{5}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = (N + 1/2)\hbar\omega. \tag{6}$$

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The last expression in (Eq.(6)) is obtained by expressing the operators $\hat{x}, \hat{p}$ appearing in $\hat{x}^2, \hat{p}^2$ in terms of $a, a^\dagger$, expanding the squares, *maintaining the order of operators carefully* and using the relations (Eq.(5)).

Eigenvalues of N are non negative

Let ν be an eigenvalue and $|\psi\rangle$ be the corresponding normalized eigenvector:

$$N|\psi\rangle = \nu|\psi\rangle. \tag{7}$$

Taking scalar product[1] with $|\psi\rangle$, we see that $\langle\psi|N|\psi\rangle = \nu$ is positive because

$$\langle\psi|N|\psi\rangle = \langle\psi|a^\dagger a|\psi\rangle = (a\psi, a\psi) = \|a\psi\|^2 \geq 0. \tag{8}$$

The operator a lowers the eigenvalues of N

Let ν be an eigenvalue of N and $|\psi\rangle$ be the corresponding eigenvector as in (Eq.(7)). The operator a act like lowering operator for the eigenvalues of N. The vector $|\phi_1\rangle$, obtained by applying a on $|\psi\rangle$, is an eigenvector of N with eigenvalue $\nu - 1$. To see this consider $N|\phi_1\rangle$

$$\begin{aligned} N|\phi_1\rangle &= Na|\psi\rangle = (aN - a)|\psi\rangle & (9)\\ &= (a\nu - a)|\psi\rangle = (\nu - 1)a|\psi\rangle & (10)\\ \therefore N|\phi_1\rangle &= (\nu - 1)|\phi_1\rangle. & (11) \end{aligned}$$

Hence $|\phi_1\rangle$, if non zero, is an eigenvector of N with eigenvalue $\nu-1$. Using this successively, we see that the result $a^n|\psi\rangle$, of applying r powers of a on $|\psi\rangle$ would give an eigenvector of N with eigenvalue $\nu - r$:

$$N\,(\,a^r|\psi\rangle\,) = (\nu - r)\,(\,a^r|\psi\rangle\,). \tag{12}$$

Now there are two possibilities: $a^r|\psi\rangle = 0$ or else if the vector $a^r|\psi\rangle$ is non zero, it is an eigenvector of N with eigenvalue $\nu - r$. since the eigenvalues of N have to be non negative, we must have

$$a^r|\psi\rangle = 0 \text{ for all } r > \nu \tag{13}$$

Taking m to be the maximum integer such that $m < \nu$ we see that

$$a^m|\psi\rangle \neq 0 \quad \text{and} \quad a^{m+1}|\psi\rangle = 0 \tag{14}$$

Using $|0\rangle$ to denote vector obtained from $(a^m|\psi\rangle)$ after normalization, we see that $|0\rangle$ is an eigenvector of N with eigenvalue 0. To see, this consider

$$N|0\rangle = a^\dagger a(a^m|\psi\rangle) = 0 = a^\dagger(a^{m+1}|\psi\rangle) = 0. \tag{15}$$

where (Eq.(14)) has been used in the last step.
To summarize, we have the results that $|0\rangle$ is a normalized eigenvector of N with zero as an eigenvalue and satisfies

$$a|0\rangle = 0. \tag{16}$$

[1] We use (ϕ, χ), as well as Dirac notation $\langle\phi|\chi\rangle$, to denote scalar product of two vectors ϕ, χ.

The eigenvalues of N are all non negative integers

We will now show that $(a^\dagger)^r \equiv |\phi_r\rangle$ raises eigenvalue of N by r units. In other words $(a^\dagger)^r|0\rangle$ is an eigenvector of N with eigenvalue r. This proof will be completed by the method of induction.

The give statement is obviously true for $m = 0$. Let us now assume the statement be true for $r = m$, *i.e.* we assume

$$N|\phi_m\rangle = m|\phi_m\rangle. \tag{17}$$

to be true and prove the statement for $r = m + 1$:

$$\begin{aligned} N|\phi_{m+1}\rangle &= N(a^\dagger)^{m+1}|0\rangle = Na^\dagger|\phi_m\rangle \\ &= (a^\dagger N + a^\dagger)|\phi_m\rangle = a^\dagger\,(m+1)|\phi_m\rangle \\ &= (m+1)(a^\dagger|\phi_m\rangle) = (m+1)|\phi_{m+1}\rangle. \end{aligned} \tag{18}$$

Therefore, $|\phi_{m+1}\rangle$ is an eigenvector of N with eigenvalue $(m + 1)$. To summarize, we have the result that the eigenvalues of N are given by $0, 1, 2, \cdots, m, \cdots$. The corresponding normalised eigenvectors will be denoted by

$$|0\rangle, |1\rangle, |2\rangle, \cdots, |m\rangle, \cdots.$$

These are also eigenvectors of $\hat{H}$, the ket $|n\rangle$ corresponds to the eigenvalue $(n + \frac{1}{2})\hbar\omega$, because

$$\hat{H}|n\rangle = (N + \tfrac{1}{2})\hbar\omega\,|n\rangle = \hbar\omega\left(N|n\rangle + \tfrac{1}{2}|n\rangle\right) = (n + \tfrac{1}{2})\hbar\omega|n\rangle. \tag{19}$$

The third postulate of quantum mechanics tells us that the allowed energies coincide with the eigenvalues of the Hamiltonian. Hence the desired energy levels of of the harmonic oscillator are

$$\boxed{E_n = (n + \tfrac{1}{2})\hbar\omega.} \tag{20}$$

Properties of eigenvectors

Let $|n\rangle$ be the normalised eigenvector of N with eigenvalue n. Action of $a^\dagger$ on $|n\rangle$ gives a vector proportional to $|n + 1\rangle$ and we write

$$a^\dagger|n\rangle = c_n|n + 1\rangle. \tag{21}$$

The constant c_n can be found by taking the inner product of vector in Eq.(21) with itself. This leads to

$$\langle n|aa^\dagger|n\rangle = |c_n|^2\langle n + 1|n + 1\rangle = |c_n|^2. \tag{22}$$

The left hand side of Eq.(22) is seen to be $(n + 1)$ by computing

$$\begin{aligned} aa^\dagger|n\rangle &= (a^\dagger a + 1)|n\rangle, \qquad (\because [a, a^\dagger] = 1) \\ &= (N + 1)|n\rangle = (n + 1)|n\rangle \qquad (23) \\ \therefore \quad \langle n|aa^\dagger|n\rangle &= n + 1 \qquad (24) \end{aligned}$$

which when used in Eq.(22) gives $(n + 1) = |c_n|^2$ and hence the result

$$\boxed{a^\dagger|n\rangle = \sqrt{(n + 1)}|n + 1\rangle}. \tag{25}$$

In a similar fashion (verify!) we get

$$\boxed{a|n\rangle = \sqrt{(n - 1)}|n - 1\rangle}. \tag{26}$$

Recursive use of Eq.(24) leads to the result that all states $|n\rangle$ can be obtained by applying powers of $a^\dagger$ on $|0\rangle$:

$$|n\rangle = \frac{1}{\sqrt{n}} a^\dagger |n-1\rangle = \frac{1}{\sqrt{n(n-1)}} (a^\dagger)^2 |n-2\rangle = \cdots = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle. \tag{27}$$

§2 Angular momentum energy eigenvalues

We will derive the eigenvalues of angular momentum operators, denoted by J_x, J_y, J_z using the commutation relations.

$$[J_x, J_y] = i\hbar J_z, \tag{28}$$
$$[J_y, J_z] = i\hbar J_x, \tag{29}$$
$$[J_z, J_x] = i\hbar J_y. \tag{30}$$

It is easy to show that $J^2 = J_x^2 + J_y^2 + J_z^2$ commutes with each of the three angular momentum operators:

$$[J^2, J_x] = 0, \qquad [J^2, J_y] = 0, \qquad [J^2, J_z] = 0. \tag{31}$$

Note that J^2 and J_z form maximal commuting set, we cannot add any other component $\hat{n}\cdot\vec{J}$ to this set. Therefore there exist a complete set of simultaneous, orthonormal eigenvectors of operators J^2 and J_z. Let us assume $|\psi\rangle$ is a simultaneous eigenvector of J^2 and J_z with eigenvalues $\lambda\hbar^2$ ad $\mu\hbar$ respectively.

$$J^2|\psi\rangle = \lambda\hbar^2, \qquad J_z|\psi\rangle = \mu\hbar|\psi\rangle. \tag{32}$$

Recall that $\hbar$ has dimensions of angular momentum, therefore λ, μ will be dimensionless numbers.

Our aim is to get restrictions on λ and μ and find their possible values. The result will apply to any set of operators obeying the commutation relations Eq.(28)-(30). Two most important techniques in this derivation will be use of positivity of operators of the form $X^\dagger X$, where X is any operator and use of ladder operators constructed out of components of angular momentum.

We introduce two operators $J_\pm = J_x \pm iJ_y$ which will serve as ladder operators and summarize important commutation relations

$$[J_+, J_-] = 2\hbar J_z \tag{33}$$
$$[J_z, J_+] = \hbar J_+ \tag{34}$$
$$[J_z, J_-] = -\hbar J_- \tag{35}$$
$$J_+J_- = J^2 - J_z^2 + \hbar J_z \tag{36}$$
$$J_-J_+ = J^2 - J_z^2 - \hbar J_z \tag{37}$$
$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 \ . \tag{38}$$

The above results are easily derived from the basic commutation relations of the angular momentum operators given above.

Before deriving the results of eigenvalues of J^2 and J_z we wish to make useful remark:

> *Let $\hat{n} = (n_1, n_2, n_3)$ be a numerical unit vector so that $n_1^2 + n_2^2 + n_3^2 = 1$. Then $\hat{n}.\vec{J}$ represents component of $\vec{J}$ along the direction $\hat{n}$. The above commutators imply that J^2 commutes with component $\hat{n}\cdot\vec{J}$ along every direction $\hat{n}$. In the sections below, we shall find the eigenvalues of J^2 and J_z. However, the conclusions about the eigenvalues of J_z will also apply to $\hat{n}\cdot\vec{J}$ for every unit vector $\hat{n}$.*

▷ Use positivity of J_+J_- and J_-J_+ to get bounds on μ

We shall now use

$$J_+J_- = J^2 - J_z^2 + \hbar J_z \quad (39)$$
$$J_-J_+ = J^2 - J_z^2 - \hbar J_z \quad (40)$$

and show that for a fixed eigenvalue λ, the eigenvalue of J_z must be bounded. Note that $J_+ = J_-^\dagger$ and consider

$$(\psi, J_+J_-\psi) = (J_-\psi, J_-\psi) = ||J_-\psi||^2 \geq 0 \ . \quad (41)$$

Using $J_+J_- = J^2 - J_z^2 + \hbar J_z$ we get

$$(\psi, J^2 - J_z^2 + \hbar J_z\psi) \geq 0 \ .$$

Since ψ is an eigenvector of J^2 and J_z we get

$$(\psi, (\lambda\hbar^2 - \mu^2\hbar^2 + \mu\hbar^2)\psi) \geq 0$$

Therefore,

$$\lambda - \mu^2 + \mu \geq 0$$

. Similarly, starting with $(\psi, J_-J_+\psi)$, we can prove that

$$\lambda - \mu^2 - \mu \geq 0 \ .$$

These two relations give $\mu^2 < \lambda$, thus absolute value of eigenvalue of J_z is bounded.

▷ J_+ is a raising operator for J_z

Action of J_+ on $|\psi>$ gives another vector $|\phi_1\rangle$

$$|\phi_1\rangle = J_+|\phi_1\rangle$$

which is also a simultaneous eigenvector of J^2 and J_z with eigenvalues $\lambda\hbar^2$ and $(\mu+1)\hbar$, respectively, i.e.,

$$J^2|\phi_1\rangle = \lambda\hbar^2|\phi_1\rangle \quad (42)$$
$$\text{and} \qquad J_z|\phi_1\rangle = (\mu+1)\hbar|\phi_1\rangle \quad (43)$$

Proof:

$$J^2|\phi_1\rangle = J^2(J_+|\psi\rangle) \quad (44)$$
$$= (J^2J_+)|\psi\rangle \qquad (\text{ because} \qquad J^2J_+ = J_+J^2) \quad (45)$$
$$= J_+(J^2|\psi\rangle) \quad (46)$$
$$= J_+(\lambda\hbar^2|\psi\rangle) \quad (47)$$
$$\text{hence} \qquad J^2|\phi_1\rangle = \lambda\hbar^2 J_+|\psi\rangle = \lambda\hbar^2]|\phi_1\rangle \quad (48)$$

Also

$$J_z|\phi_1\rangle = J_zJ_+|\psi\rangle \qquad (\text{use} \qquad J_zJ_+ - J_+J_z = \hbar J_+) \quad (49)$$
$$= (J_+J_z + \hbar J_+)|\psi> \quad (50)$$
$$= (J_+\mu\hbar + \hbar J_+)|\psi> \quad (51)$$
$$= (\mu+1)\hbar J_+|\psi>= (\mu+1)\hbar|\phi_1\rangle \quad (52)$$
$$J_z|\phi_1\rangle = (\mu+1)\hbar|\phi_1\rangle \quad (53)$$

Similarly, the states $|\phi_2\rangle = J_+|\phi_1\rangle, |\phi_3\rangle = J_+|\phi_2\rangle \cdots$, obtained by repeated action of J_+ on $|\psi\rangle$, are eigenvectors of J^2 and J_z as given below.

States	Eigenvalue of J^2	Eigenvalue of J_z
$\|\phi_1\rangle = J_+\|\psi\rangle$	$\lambda\hbar^2$	$(\mu+1)\hbar$
$\|\phi_2\rangle = J_+\|\phi_1\rangle = J_+^2\|\psi\rangle$	$\lambda\hbar^2$	$(\mu+2)\hbar$
$\|\phi_3\rangle = J_+\|\phi_2\rangle = J_+^3\|\psi\rangle$	$\lambda\hbar^2$	$(\mu+3)\hbar$
$\cdots$	$\cdots$	$\cdots$

J_- is a lowering operator for J_z In a manner similar to the step II above, the commutation relation

$$J_zJ_- - J_-J_z = -\hbar J_-$$

or

$$J_zJ_- = J_-(J_z - \hbar)$$

can be used to show that the states $|\chi_1\rangle, |\chi_2\rangle, \cdots$, obtained by repeated application of J_- on $|\psi\rangle$ are also eigenvectors of J^2 and J_z with bar eigenvalues as given below

e States	Eigenvalue of J^2	Eigenvalue of J_z
$\|\chi_1\rangle = J_-\|\psi\rangle$	$\lambda\hbar^2$	$(\mu-1)\hbar$
$\|\chi_2\rangle = J_-\|\chi_1\rangle = (J_-)^2\|\psi\rangle$	$\lambda\hbar^2$	$(\mu-2)\hbar$
$\|\chi_3\rangle = J_-\|\chi_2\rangle = (J_-)^3\|\chi_3\rangle$	$\lambda\hbar^2$	$(\mu-3)\hbar$
$\cdots$	$\cdots$	$\cdots$

Thus if μ is an eigenvalues of j_z, then $(\mu\pm1)\hbar, (\mu\pm2)\hbar, (\mu\pm3)\hbar\cdots$ are all eigenvalues of J_z. Thus statement holds as long as an application of J_+ (or J_-) gives a non-zero vector.

▷ The difference between the maximum and minimum values of μ is an integer

We have already seen that the eigenvalues λ and μ satisfy

$$\mu^2 \leq \lambda$$

This relation show that for a fixed λ the eigenvalue μ of J_z cannot increase or decrease indefinitely. Thus, for a given value of λ, there is a maximum eigenvalue and there is a minimum eigenvalue of J_z.

Let μ_1 be the minimum eigenvalue of J_z for given λ and let $|\chi\rangle$ be corresponding eigenvector

$$J^2|\chi\rangle = \lambda\hbar^2|\chi\rangle \qquad J_z|\chi\rangle = \mu_1\hbar|\chi\rangle$$

Then action of J_- on $|\chi\rangle$ must give null vector, otherwise $J_-|\chi\rangle$ will be eigenvector of J^2 and J_z with eigenvalues $\lambda\hbar^2$ and $(\mu_1-1)\hbar$ respectively and this would contradict the result that μ_1 is the minimum eigenvalue of J_z (for fixed λ). Hence

$$J_-|\chi\rangle = 0\ \ .$$

Taking norm of $J_-|\chi\rangle$ we get

$$\langle\chi|J_+J_-|\chi\rangle = 0$$

or

$$\langle\chi|J^2 - J_z^2 + \hbar J_z|\chi\rangle = 0$$

or

$$\lambda\hbar^2 - \mu_1^2\hbar^2 + \mu_1\hbar^2 = 0$$

or

$$\lambda - \mu_1^2 + \mu_1 = 0$$

Starting with $|\chi\rangle$, we can generate eigenvectors of J_z by repeated application of J_+ and we shall get values $(\mu_1+1)\hbar, (\mu_1+2)\hbar, \cdots, (\mu_1+n)\hbar$. Let μ_2 be the maximum allowed value.

Let $|\phi\rangle$ be eigenvector of J_z with eigenvalue μ_2 which is the maximum allowed value for a given λ. Then
(a) $J_+|\phi\rangle = 0 \qquad J_z|\phi\rangle = \mu_2\hbar|\phi\rangle$
(b) $\mu_2 - \mu_1 =$ an integer M.
Taking norm of $J_+|\phi\rangle$ we get

$$\langle\phi|J_-J_+|\phi\rangle = 0 \quad \text{or} \quad \langle\phi|J^2 - J_z^2 - \hbar J_z|\phi\rangle = 0$$

or

$$\lambda\hbar^2 - \mu_2^2\hbar^2 - \mu_2\hbar^2 = 0$$

Therefore we have

$$\lambda - \mu_2^2 - \mu_2 = 0$$

The above equation and corresponding equation for μ_1

$$\lambda - \mu_1^2 + \mu_1 = 0$$

imply that

$$\mu_2^2 + \mu_2 - \mu_1^2 + \mu_1 = 0 \tag{54}$$
$$\text{or} \quad (\mu_2 - \mu_1)(\mu_2 + \mu_1) + (\mu_2 + \mu_1) = 0 \tag{55}$$
$$\text{or} \quad (\mu_2 + \mu_1)(\mu_2 - \mu_1 + 1) = 0 \tag{56}$$
$$\mu_2 = -\mu_1 \qquad \text{or} \qquad \mu_2 = \mu_1 - 1 \tag{57}$$

Since $\mu_2 = \mu_1 - 1 \Longrightarrow \mu_2 < \mu_1$ which contradicts our earlier assumptions that μ_2 is the maximum and μ_1 is the minimum allowed eigenvalue of J_z, hence we must have

$$\mu_2 = -\mu_1$$

This equation, when used with $\mu_2 - \mu_1 = M$, implies that

$$\mu_2 = -\mu_1 = \frac{M}{2} \ . \tag{58}$$
$$\text{and} \quad \lambda = \mu_2(\mu_2 + 1) = \frac{M}{2}\left(\frac{M}{2} + 1\right) \tag{59}$$

Next we change notation and call $\frac{M}{2} = j$ then the eigenvalues of J^2 are

$$J_z^2 \to j(j+1)\hbar^2 \qquad \text{with} \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

and, for a fixed value of j, the eigenvalues of J_z are between $\mu_2\hbar = (M/2)\hbar = j\hbar$ and $\mu_1\hbar = -M/2\hbar = -j\hbar$

▷ The results

Therefore, J_z has $(2j+1)$ eigenvalues

$$-j\hbar, (-j+1)\hbar, \cdots, (j-1)\hbar, j\hbar$$

for the square of the total angular momentum, J^2, having value $j(j+1)\hbar^2$ and the possible values of j are $j = 0, 1/2, 1, 3/2, \cdots$

▷ A few useful results

1. The following results turn out to be extremely useful

$$\begin{aligned} J_+|jm\rangle &= \sqrt{j(j+1) - m(m+1)}\hbar|j, m+1\rangle & (60) \\ J_-|jm\rangle &= \sqrt{j(j+1) - m(m-1)}\hbar|j, m-1\rangle & (61) \end{aligned}$$

2. We add special cases of these results for $m = \pm j$.

$$\begin{aligned} J_+|jm = j\rangle &= 0 & (62) \\ J_-|jm = -j\rangle &= 0 & (63) \end{aligned}$$

We will now prove the above relations. We have seen that the operators $J_\pm$ act as ladder operators, acting on a ket $|jm\rangle$ these operators give a state proportional to $|j, m \pm 1\rangle$ so that

$$J_+|jm\rangle = C|j, m+1\rangle \quad (64)$$

Taking scalar product of this equation with itself we get

$$\langle jm|J_-J_+|jm\rangle = |C|^2\langle j, m+1|j, m+1\rangle \quad (65)$$

Using the relation $J_-J_+ = J^2 - J_z^2 - \hbar J_z$ and the fact that $|jm\rangle$ and $|jm+1\rangle$ are normalized we get

$$|C|^2 = j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2 \quad (66)$$

Thus we have

$$\begin{aligned} J_+|jm\rangle &= \sqrt{j(j+1) - m(m+1)}|jm+1\rangle & (67) \\ J_-|jm\rangle &= \sqrt{j(j+1) - m(m-1)}|jm-1\rangle & (68) \end{aligned}$$

▷ Summary

1. The angular momentum commutation relations are

$$[J_x, J_y] = i\hbar J_z; \quad (69)$$
$$[J_y, J_z] = i\hbar J_x; \quad (70)$$
$$[J_z, J_x] = i\hbar J_y; \quad (71)$$

2. It is useful to introduce the operators $J_\pm = J_x \pm iJ_y$ and these operators obey commutation relations

$$\begin{aligned} [J_+, J_-] &= 2\hbar J_z & (72) \\ [J_z, J_+] &= \hbar J_+ & (73) \\ [J_z, J_-] &= -\hbar J_- & (74) \end{aligned}$$

   Also $\vec{J}^2$ commute with all the three components of the angular momentum J_x, J_y, J_z.

3. Some useful relations are

$$J_+J_- = J^2 - J_z^2 + \hbar J_z \tag{75}$$
$$J_-J_+ = J^2 - J_z^2 - \hbar J_z \tag{76}$$
$$J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2 \ . \tag{77}$$

4. One can find simultaneous eigenvectors of $\vec{J}^2$ and *any one* component, $J_n = \hat{n} \cdot \vec{J}$, along a fixed direction given by the unit vector $\hat{n}$.

5. The eigenvalues of $\vec{J}^2$ are $j(j+1)\hbar^2$ where j can be integer of half integer. The eigenvalues of a component of J_n take values from $-j$ to j in step of 1.

6. It is cutstomary to denote the *normalized*, simultaneous,eigenvectors of $\vec{J}^2$ and J_z by $|jm\rangle$ so that

$$\vec{J}^2|jm\rangle = j(j+1)\hbar|jm\rangle \tag{78}$$
$$J_z|jm\rangle = m\hbar|jm\rangle \tag{79}$$

7. The operators $J_\pm$ acting on $|jm\rangle$ give a ket vector *proportional* to $|jm \pm 1\rangle$. The proportionality coefficient can be worked out using the relation Eq.(36) and Eq.(76). One then gets

$$J_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, \hbar\, |jm \pm 1\rangle \tag{80}$$

8. The operators $J_\pm$ annihilate $|j, \pm j\rangle$ because the J_z value cannot be increased beyond j nor can it decrease beyond $-j$.

$$J_+|j, j\rangle = 0 \qquad\qquad J_-|j, -j\rangle = 0 \tag{81}$$

9. The results derived for eigenvalues of J_z also hold for any component of $\vec{J}$. Thus, when $j(j+1)\hbar^2$ is eigenvalue of $\vec{J}^2$, for every unit vector $\hat{n}$, the eigenvalues of $\hat{n} \cdot \vec{J}$ are $-j, -j+1, \ldots, j$.

10. The above results are applicable to operators satisfying angular momentum commutation relations except that the half integral values are ruled out for the orbital angular momentum, because of additional requirement of single valuedness of the wave function. This last statement cannot be proved using algebra of angular momentum operators alone, to derive it one has to work in coordinate representation and demand single valuedness of the eigenfunctions of L^2, L_z.