Existence of Series Solution

Fuch's Theorem

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Abstract

The conditions for existence of series solution and analytic properties of the series solution (Fuch's Theorem) are described.

FROBENIUS METHOD

The Frobenius method of series solution is a useful method for a large class of linear ordinary differential equations of mathematical physics. A general ordinary second order linear differential equation can be put in the form

$$\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + q(x)y(x) = 0 \tag{1}$$

In the Frobenius method of series solution, it is assumed that the solution can be written in the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} = x^c \left[a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots\right] \tag{2}$$

The parameter c is called ***index***. The expansion parameters a_n and the index c are determined by substituting Eq.(2) in the ODE Eq.(1) , expanding $p(x)$ and $q(x)$ in powers of x, and comparing the coefficients of different powers of x on both sides. The coefficient of the general power n equated to zero gives recurrence relations for the coefficients of expansion a_n. These recurrence relations are then solved and the expansion coefficients are fixed.
When this method is applicable, one gets two linearly independent solutions $y_1(x)$ and $y_2(x)$ for the second order differential equations. The most general solution $y(x)$ of the ODE Eq.(1) is then represented as a linearly combination of the solutions $y_1(x)$ and $y_2(x)$.

$$y_1(x) = \alpha y_1(x) + \beta y_2(x) \tag{3}$$

where the constants α and β are to be fixed by initial conditions. It must be remarked that the two linear independent solutions are not always of the form Eq.(2) assumed in the beginning . In general one may also get a series of type Eq.(2) multiplied by $\log x$. The expansion Eq.(2) is expansion about the point $x = 0$. In general one may attempt a series solution about any point x_0. In such a case, instead of Eq.(2) , one assumes the solution to be of the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n (x - x_0)^c + n \tag{4}$$

We now summarize the method of obtaining two linearly independent solutions in the four cases of series solution.

CASE I:

In this case the roots of the indicial equation are distinct and the difference of the roots c_1 and c_2 is not an integer. The two linearly independent solutions are given by

$$y_1(x) = y(x,c)|_{c=c_1} and y_2(x) = y(x,c)|_{c=c_2}$$

CASE II:

In this case the roots of the indicial equation are equal to, say, c_0. The two linearly independent solutions are given by

$$y_1(x) = y(x,c)|_{c=c_0} \text{and} y_2(x) = \left.\frac{\mathrm{d}}{\mathrm{dc}} y(x,c)\right|_{c=c_0}$$

CASE III:

In this case the roots of the indicial equation, c_1 and c_2 is an integer.And one of the coefficients becomes infinite for one of the values of c, which we assume to be c_1. In this case we assume $control a_0 = k(c - c_1), \quad k \neq 0$. The two linearly independent solutions are then given by

$$y_1(x) = y(x,c)|_{c=c_1} \text{and} y_2(x) = \left.\frac{\mathrm{d}}{\mathrm{dc}} y(x,c)\right|_{c=c_1}$$

The solution obtained from $y(x,c)$ by setting $c = c_2$ is identical with $y_1(x)$ apart from an over all constant.

CASE IV:

In this case the roots of the indicial equation, c_1 and c_2 are distinct and the difference of the roots c_1 and c_2 is an integer. And one of the coefficients, say a_{n}, becomes indeterminate for one of the values of c, which we assume to be c_1. In this case we keep a_0 and a_{n} as unknown constants, and the most general solution containing two unknown constants is obtained from $y(x,c)$ setting $c = c_1$.

$$y(x) = y(x,c)|_{c=c_1}$$

The solution obtained from $y(x,c)$ by setting $c = c_2$ coincides with $y(x)$ for particular values of the constants a_0 and a_{n}.

§0.1 Convergence of Series Solutions

We are now interested in knowing the properties of the solutions.
Having obtained the solutions in a series form one must ask what are the values of x for which the series appearing in the solutions converge? When do we have two linearly independent solutions?
The answer to these and related questions is given by Fuch's Theorem.For this purpose it turns out to be useful to regard the independent variable x as complex variable and to continue the two functions $p(x)$ and $q(x)$ to the complex plane.
As a preparation to the statement of the Fuch's Theorem we define an ordinary point, the regular singular and the irregular singular points of an ordinary differential equation. As already mentioned the independent variable x will regarded as a complex variable.

A point $x = x_0$ in the complex plane is called an ***ordinary point*** of the differential equation if both the functions $p(x)$ and $q(x)$ are analytic at $x = x_0$.

A point x_0 is called ***singular point*** of the ordinary differential equation, if it is not an ordinary point.

A singular point x_0 is called ***regular singular point*** of the differential equation if the two functions $P(x)$ and $Q(x)$, where

$$P(x) = (x - x_0)p(x), \qquad Q(x) = (x - x_0)^2 q(x) \tag{5}$$

are analytic at x_0.
A singular point x_0 is called ***irregular singular point*** if it is not a regular singular point.

Point at Infinity

We say that the point at infinity ($x = \infty$) is, respectively, an ordinary point, a regular singular point of a differential equation Eq.(1) if for the corresponding equation Eq.(5) ,in $t = 1/x$, the point $t = 0$ is an ordinary point or a regular singular point. Asimilar statement holds for the irregular singular points.

Theorem **1** *If x_0 is an ordinary point of the differential equation Eq.(1) , there exist two linearly independent solutions which are analytic at x_0. These solutions are therefore expressible as power series in $(x - x_0)$ in the form Eq.(4) . The radius of convergence of the power series is at least as large as the distance of x_0 from the nearest singular point of the functions $p(x)$ and $q(x)$ in the complex plane.*

The Fuch's theorem given below summarises the corresponding results for the series solution about a regular singular point.

Theorem **2 (Fuch's Theorem)** *If the differential equation Eq.(1) has a regular singular singular point at $x = x_0$ there exist two linearly independent solutions which can be expressed in the form*

$$y(x) = (x - x_0)^c[\log(x - x_0)\phi_1(x) + \phi_2(x)] \tag{6}$$

where $\phi_1(x)$ and $\phi_2(x)$ have power series expansions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{7}$$

The series expansions for $\phi_1(x)$ and $\phi_2(x)$ have radius of convergence at least as large as the distance of x_0 from the nearest point, in the complex plane, of $P(x)$ and $Q(x)$ as defined in Eq.(5) .