Case IV: Roots of indicial equation differ by an integer

Some Coefficients Remain Undetermined

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Abstract

The method of series solution explained for a differential equation which has distinct roots of indicial equation, differing by an integer, and some coefficient becomes indeterminate.

In this lecture we shall take up solution of an ordinary differential equation by the method of series solution. The example to be discussed is such that the difference of the roots of the indicial equation is an integer and some coefficient becomes indeterminate.

Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2 y = 0 \tag{1}$$

Substituting

$$y = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{2}$$

in Eq.(1) we get

$$\sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} + x^2 \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{3}$$

or,

$$\sum_{n=0}^{\infty} a_n (n+c)(n+c-1) x^{n+c-2} + \sum_{n=0}^{\infty} a_n x^{n+c+2} = 0 \tag{4}$$

The lowest power of x in the right hand side of Eq.(4) is x^{c-2}. This gives

$$a_0 c(c-1) = 0 \tag{5}$$

Therefore the two values of c are $c = 0$ and $c = 1$. Equating the coefficients of $x^{c-1}, x^c, x^{c+1}, x^{c+2}, \ldots$ to zero successively gives

$$a_1 c(c+1) = 0, \tag{6}$$

$$a_2 (c+1)(c+2) = 0, \tag{7}$$

$$a_3 (c+2)(c+3) = 0, \tag{8}$$

$$a_4 (c+4)(c+3) + a_0 = 0. \tag{9}$$

The recurrence relation obtained by considering the coefficient of x^{m+c+2} is

$$a_{m+4}(c+m+4)(c+m+3) + a_m = 0. \tag{10}$$

The solution for $c = 1$ can be constructed easily using the recurrence relations.

Let us now look at the case $c = 0$. In this case, from Eq.(6) we get

$$a_1 .0 = 0. \tag{11}$$

Thus a_1 cannot be fixed and is indeterminate. In this case we proceed as before except that we retain both a_0 and a_1 as unknown parameters. We construct solution for this case, $c = 0$, first and then come back and look at the solution for $c = 1$.

Case $c = 0$:

Substituting $c = 0$ from Eq.(6) to Eq.(10) we get

$$a_2 = a_3 = 0; \quad a_4 = -\frac{a_0}{4.3} \tag{12}$$

$$a_{m+4} = -\frac{a_m}{(m+4)(m+3)} \tag{13}$$

Combining Eq.(12) and Eq.(13) we see that

$$a_2 = a_6 = a_1 0 \cdots 0 \tag{14}$$

and

$$a_3 = a_7 = a_1 1 \cdots 0 \tag{15}$$

Also

$$a_4 = -\frac{1}{4.3}a_0; \quad a_8 = -\frac{1}{8.7}a_4; \quad a_{12} = -\frac{1}{12.11}a_8 \tag{16}$$

$$a_5 = -\frac{1}{5.4}a_1; \quad a_9 = -\frac{1}{9.8}a_5; \quad a_{13} = -\frac{1}{13.12}a_9 \tag{17}$$

Solving Eq.(16) and Eq.(17) we get

$$a_4 = -\frac{1}{4.3}a_0; \quad a_8 = -\frac{1}{8.7.4.3}a_0; \quad a_{12} = -\frac{1}{12.11.8.7.4.3}a_0 \tag{18}$$

$$a_5 = -\frac{1}{5.4}a_1; \quad a_9 = -\frac{1}{9.8.5.4}a_1; \quad a_{13} = -\frac{1}{13.12.9.8.5.4}a_1 \tag{19}$$

The series solution in this case contains two parameters, which are not determined by the recurrence relations, and is given by

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{20}$$

$$y_1(x) = 1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \frac{x^{12}}{3.4.7.8.11.12} + \cdots \tag{21}$$

$$y_2(x) = x\left\{1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \frac{x^{12}}{4.5.8.9.12.13} + \cdots\right\} \tag{22}$$

These two functions $y_1(x)$ and $y_2(x)$ represent two linearly independent solutions. What happens when one tries to construct the solution for the second value of c ? In this case we recover one of the above two solutions already obtained. This will now be demonstrated explicitly.

Case $c = 1$:

In this case we get

$$a_1 = a_2 = a_3 = 0 \tag{23}$$

$$a_{m+4} = -\frac{a_m}{(m+5)(m+4)} \tag{24}$$

We therefore get

$$a_4 = -\frac{1}{5.4}a_0; \quad a_8 = -\frac{1}{9.8}a_4; \quad a_{12} = -\frac{1}{13.12}a_8 \tag{25}$$

Compare the equations Eq.(25) with Eq.(17) . We now construct the series

$$y = x^c \sum a_n x^n \tag{26}$$

and get

$$y_2(x) = a_0 x\left\{1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \frac{x^{12}}{4.5.8.9.12.13} + \cdots\right\} \tag{27}$$

This solution coincides with $y_2(x)$ of Eq.(22) except for an overall constant. Hence the most general solution of the differential equation Eq.(1) is given by Eq.(20) .