Equations with Constant Coefficients

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Abstract

An example of solution of a second order differential equation by the method of series solution is presented. This case corresponds to the case when the two solutions of the indicial equation do not differ by an integer. In this case the second solution is obtained in a straight forward manner. does not require any extra steps to construct the second solution.

An n^{th} order ordinary differential equation has the form

$$a_0(x)\frac{d^n y(x)}{dx^n} + a_1(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y(x)}{dx^{n-2}} + \ldots + a_n(x)y(x) = P(x) \qquad (1)$$

where the coefficients $a_0(x), a_1(x), a_2(x), \ldots, a_n(x)$ are in general functions of x. However there are two special cases of interest.

CASE I : $a_0, a_1, a_2, \ldots, a_n$ are constants independent of x and $a_0 \neq 0$ In this case we say that the differential equation is the n^{th} order linear equation with constant coefficients.

CASE II : $a_j(x)$ are proportional to x^{n-j}.In this case the equation is known as the Euler equation.

In both these cases the complete solution of the differential equation can be written down .

Case I :Constant Coefficients

Ordinary differential equation of n^{th} order with constant coefficients have the form $[a_0 = 1]$

$$\frac{d^n y(x)}{dx^n} + a_1\frac{d^{n-1}y(x)}{dx^{n-1}} + a_2\frac{d^{n-2}y(x)}{dx^{n-2}} + \ldots + a_n y(x) = 0 \qquad (2)$$

Where $a_1, a_2, \ldots, a_n$ are constants. Denoting the differential operator in the left hand side as L where

$$L = \frac{d^n}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + a_2\frac{d^{n-2}}{dx^{n-2}} \ldots + a_n \qquad (3)$$

This equation can be solved by taking a trial solution of the form

$$y(x, \lambda) = \exp[\lambda x] \qquad (4)$$

computing

$$Ly(x, \lambda) = [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n\ ]y(x, \lambda) \qquad (5)$$

We see that $Ly = 0$ if λ is a root of the equation

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0 \tag{6}$$

the r.h.s of Eq.(5) will become zero and corresponding $y(x, \lambda)$ will be a solution. Hence we know that if the Eq.(6)has n distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the ODE Eq.(2) has n solutions.

$$y_1(x) = e^{\lambda_1 x};\ \ y_2(x) = e^{\lambda_2 x};\ \ldots\ y_n(x) = e^{\lambda_n x}; \tag{7}$$

WHAT IF SOME ROOTS OF THE EQUATION Eq.() HAVE MULTIPLICITIES GREATER THAN 1 ?

Let us consider a simple concrete example of a second order differential equation

$$\frac{d^2y(x,\lambda)}{dx^2} - 2\alpha\frac{dy(x,\lambda)}{dx} + \alpha^2 y(x) = 0 \tag{8}$$

$y(x, \lambda)$ is a solution if

$$\lambda^2 - 2\alpha\lambda + \alpha^2 = 0$$

This equation has a double root $\lambda = \alpha$. This gives one solution $y(x) = e^{\alpha x}$.
There is a second solution which can be found by several methods.

METHOD 1:

Substituting $y(x, \lambda) = e^{\lambda x}$ for $y(x)$ in Eq.(8) we get,[compare with Eq.(5)]

$$\frac{d^2y(x,\lambda)}{dx^2} - 2\alpha\frac{dy(x,\lambda)}{dx} + \alpha^2 y(x,\lambda) = (\lambda - \alpha)^2 y(x,\lambda) \tag{9}$$

Note that not only the right hand side vanishes for $\lambda = \alpha$, also the first derivative of right hand side vanishes w.r.t λ for $\lambda = \alpha$. Thus

$$\left[\frac{d^2}{dx^2} - 2\alpha\frac{d}{dx} + \alpha^2\right] y(x,\lambda)\bigg|_{\lambda=\alpha} = 0 \tag{10}$$

Since the order of derivatives w.r.t λ and w.r.t x can be interchanged, the Eq.(10) is equivalent to

$$\left[\frac{d^2}{dx^2} - 2\alpha\frac{d}{dx} + \alpha^2\right] \frac{d}{d\lambda} y(x,\lambda)\bigg|_{\lambda=\alpha} = 0 \tag{11}$$

Thus $\frac{d}{d\lambda}y(x, \lambda)$ is a solution of the given differential equation for $\lambda = \alpha$.
This gives the second solution as $y_2(x) = xe^{\alpha x}$

METHOD 2:

Suppose we start from a differential equation

$$\frac{d^2y(x,\lambda)}{dx^2} - (\alpha+\beta)\frac{dy(x,\lambda)}{dx} + \alpha\beta y(x,\lambda) = 0 \tag{12}$$

Which has two distinct solutions

$$y_1(x) = e^{\alpha x};\ y_2(x) = e^{\beta x} \tag{13}$$

We then ask what happens when β tends to α ? Obviously the second solution $y_2(x)$ tends to the first solution $y_1(x)$ and the two solutions 13 are no longer independent. However, we can make use of the fact that 12 is a linear differential equation and any superposition of two solutions is also a solution. Thus we may write

$$y_3(x) = Ay_1(x) + By_2(x) \tag{14}$$

and select A and B in such a way that even in the limit $\beta \longrightarrow \alpha, y_3(x)$ remains independent of $y_1(x)$ and $y_2(x)$. One possible choice of A and B having this property is $A = 1/(\alpha - \beta)$ and $B = 1/(\alpha - \beta)$ .With this choice 14 becomes

$$y_3(x) = \frac{y_1(x) - y_2(x)}{\alpha - \beta} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} \tag{15}$$

and in the limit $\alpha \longrightarrow \beta$ Eq.(14) tends to the desired solution $xe^{(\alpha x)}$!

METHOD 3:

There is yet one more method, called the method of variation of constants which gives a second solution directly in terms of the first solution .We shall show how this method works for more general ordinary differential equations. After obtaining the final result we shall apply it to the case of the example Eq.(8) for which one solution $y_1(x) = e^{\alpha x}$ is already known.
In this method one writes the second solution as

$$y(x) = u(x)y_1(x) \tag{16}$$

and demand that the differential equation satisfied by $u(x)$ be of one order lower. In this example equation for u will of order 1.
Let $y_1(x)$ be a solution of the equation

$$\left[\frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x)\right] y(x) = 0 \tag{17}$$

substituting Eq.(16) in Eq.(17) we get

$$\frac{d^2[u(x)y_1(x)]}{dx^2} + a(x)\frac{d[u(x)y_1(x)]}{dx} + b(x)u(x)y_1(x) = 0 \tag{18}$$

Using the fact that $y_1(x)$ satisfies the original equation Eq.(17) we get an equation for $u(x)$ as

$$y_1(x)\frac{d^2u(x)}{dx^2} + a(x)y_1(x)\frac{du(x)}{dx} + 2\frac{dy_1(x)}{dx}\frac{du(x)}{dx} = 0 \tag{19}$$

writing $v(x) = \frac{du(x)}{dx}$ the Eq.(19) takes the form

$$y_1(x)\frac{dv(x)}{dx} + a(x)y_1(x)v(x) + 2\frac{dy_1(x)}{dx}v(x) = 0 \tag{20}$$

This equation is of first order and can be solved for $v(x)$, this solution in turn gives $u(x)$.
To solve Eq.(20) we multiply it by $y_1(x)$ and rearrange in the form

$$\frac{d}{dx}\left[y_1^2(x)v(x)\right] = -a(x)y_1^2(x)v(x) \tag{21}$$

To solve the above equation define $w(x) \equiv y_1^2(x)v(x)$ and solve for $w(x)$ .We thus get

$$\frac{1}{w(x)}\frac{dw(x)}{dx} = -a(x) \tag{22}$$

Hence

$$w(x) = ce^{[-\int^x a(t)dt]} \tag{23}$$

Substituting $w(x) = y_1^2(x)v(x) = y_1^2(x)\frac{du(x)}{dx}$, and solving for u(x)we get

$$u(x) = c\int \frac{1}{y_1^2(x)} e^{[-\int^x a(t)dt]} dx \tag{24}$$

$$y_2(x) = y_1(x)u(x) = cy_1(x)\int \frac{1}{y_1^2(x)} e^{[-\int^x a(t)dt]} dx \tag{25}$$

Eq.(8) is a special case of Eq.(17) with $a(x) - -2\alpha$, $b(x) = \alpha^2$ and the one known solution is $y_1(x) = e^{-\alpha x}$. Making these substitutions in Eq.(25) the second solution is easily computed to be

$$y_2(x) = cxy_1(x) \tag{26}$$

This coincides with the standard known solution. In general, $y_2(x)$ so obtained will be a linear combination of solutions obtained by other methods. All the three methods can be generalized to include cases of higher multiplicities and higher order differential equations.