Lecture Notes
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Ordinary Differential Equations

A.K. Kapoor
http://0space.org/users/kapoor
akkhcu@gmail.com; kapoor.proofs@gmail.com

Contents

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Resources and Acknowledgements

These content of these lecture notes was delivered as part of Mathematical Physics course in the M.Sc. program of University of Hyderabad. These lecture notes borrow heavily from Piaggio[**?**]. Other useful references are Rabenstein [**?**] and Simmons [**?**], Dennery and Kryzwicki [**?**]

For a readable and detailed account of existence and uniqueness theorems Coddington[**?**] is recommended.

For detailed and rigorous mathematical treatment of various aspects of ordinary differential equations see Ince[**?**] and Coddington and Levinson. [**?**].

An extremely useful reference is the book by Tulsi Dass and S. K. Sharma [**?**] where proofs of convergence of series solution can be found.

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§1 Equations With Constant Coefficients

An n^{th} order ordinary differential equation has the form

$$a_0(x)\frac{d^n y(x)}{dx^n}+a_1(x)\frac{d^{n-1}y(x)}{dx^{n-1}}+a_2(x)\frac{d^{n-2}y(x)}{dx^{n-2}}+\ldots+a_n(x)y(x) = P(x) \tag{1}$$

where the coefficients $a_0(x), a_1(x), a_2(x), \ldots, a_n(x)$ are in general functions of x. There are two special cases of interest.

CASE I : $a_0, a_1, a_2, \ldots, a_n$ are constants independent of x and $a_0 \neq 0$ In this case we say that the differential equation is the n^{th} order linear equation with constant coefficients.

CASE II : $a_j(x)$ are proportional to x^{n-j}.In this case the equation is known as the Euler equation.

In both these cases the complete solution of the differential equation can be written down .

Case I: Equations with constant coefficients

Ordinary differential equation of n^{th} order with constant coefficients have the form $[a_0 = 1]$

$$\frac{d^n y(x)}{dx^n} + a_1\frac{d^{n-1}y(x)}{dx^{n-1}} + a_2\frac{d^{n-2}y(x)}{dx^{n-2}} + \ldots + a_n y(x) = 0, \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are constants. Writing the differential equation as $Ly = 0$ where L is the differential operator in the left hand side of (2)

$$L = \frac{d^n}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + a_2\frac{d^{n-2}}{dx^{n-2}} \ldots + a_n. \tag{3}$$

This equation can be solved by taking a trial solution of the form

$$y(x, \lambda) = \exp[\lambda x] \tag{4}$$

computing

$$Ly(x, \lambda) = [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n\,]y(x, \lambda) \tag{5}$$

We see that $Ly = 0$ is satisfied by the trial solution if λ is a root of the equation

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0. \tag{6}$$

the r.h.s of Eq.(5) will become zero and corresponding $y(x, \lambda)$ will be a solution. Hence we know that if the Eq.(6) has n distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the ODE Eq.(2) has n solutions.

$$y_1(x) = e^{\lambda_1 x};\;\; y_2(x) = e^{\lambda_2 x};\;\; \ldots\;\; y_n(x) = e^{\lambda_n x}. \tag{7}$$

WHAT IF SOME ROOTS OF THE EQUATION Eq.(6) HAVE MULTIPLICITIES GREATER THAN 1 ?

Let us consider a simple concrete example of a second order differential equation

$$Ly(x) \equiv \frac{d^2y(x,\lambda)}{dx^2} - 2\alpha\frac{dy(x,\lambda)}{dx} + \alpha^2 y(x) = 0. \tag{8}$$

The trial function $y(x, \lambda) = \exp(\lambda x)$ is a solution, if

$$\lambda^2 - 2\alpha\lambda + \alpha^2 = 0.$$

This equation has a double root $\lambda = \alpha$. This gives one solution $y(x) = e^{\alpha x}$.
There is a second solution which can be found by several methods discussed below.

Method 1:

Substituting $y(x, \lambda) = e^{\lambda x}$ in Eq.(8)for $y(x)$ in Eq.(8) and computing $Ly(x, \lambda)$ we get,[compare with Eq.(5)]

$$\frac{d^2y(x,\lambda)}{dx^2} - 2\alpha\frac{dy(x,\lambda)}{dx} + \alpha^2 y(x,\lambda) = (\lambda-\alpha)^2 y(x,\lambda) \tag{9}$$

Note that the right hand side of (9) vanishes for $\lambda = \alpha$, giving the first solution as$y_1(x) = y(x, \alpha$. Also note that the first derivative of right hand side w.r.t. λ vanishes for $\lambda = \alpha$. Thus differentiating Eq.(9) w.r.t. λ and setting $\lambda = \alpha$ we get

$$\frac{d}{d\lambda}\left[\frac{d^2}{dx^2} - 2\alpha\frac{d}{dx} + \alpha^2\right] y(x,\lambda)\bigg|_{\lambda=\alpha} = 0. \tag{10}$$

Since the order of derivatives w.r.t λ and w.r.t x can be interchanged, the Eq.(10) is equivalent to

$$\left[\frac{d^2}{dx^2} - 2\alpha\frac{d}{dx} + \alpha^2\right]\frac{d}{d\lambda}y(x,\lambda)\bigg|_{\lambda=\alpha} = 0 \tag{11}$$

Thus $\frac{d}{d\lambda}y(x,\lambda)$ is a solution of the given differential equation for $\lambda = \alpha$.
This gives the second solution as $y_2(x) = \left(\frac{d}{d\lambda}e^{\lambda x}\right)\big|_{\lambda=\alpha} = xe^{\alpha x}$.

Method 2:

Suppose we start from a differential equation

$$\frac{d^2y(x,\lambda)}{dx^2} - (\alpha+\beta)\frac{dy(x,\lambda)}{dx} + \alpha\beta y(x,\lambda) = 0 \tag{12}$$

which has two distinct solutions

$$y_1(x) = e^{\alpha x}; \qquad y_2(x) = e^{\beta x}. \tag{13}$$

We then ask what happens when β tends to α? Obviously, the second solution $y_2(x)$ tends to the first solution $y_1(x)$ and the two solutions (13) are no longer independent. However, we can make use of the fact that (12) is a linear differential equation and any superposition of two solutions is also a solution. Thus we may write

$$y_3(x) = Ay_1(x) + By_2(x) \tag{14}$$

and select A and B in such a way that even in the limit $\beta \longrightarrow \alpha, y_3(x)$ remains independent of $y_1(x)$ and $y_2(x)$. One possible choice of A and B having this property is $A = 1/(\alpha-\beta)$ and $B = -1/(\alpha-\beta)$ .With this choice Eq.(14) becomes

$$y_3(x) = \frac{y_1(x) - y_2(x)}{\alpha-\beta} = \frac{e^{\alpha x} - e^{\beta x}}{\alpha-\beta} \tag{15}$$

and in the limit $\alpha \to \beta$ Eq.(15) tends to the desired solution:

$$\lim_{\beta\to\alpha} y_3(x) = \lim_{\beta\to\alpha}\frac{e^{\alpha x} - e^{\beta x}}{\alpha-\beta} = \frac{d}{d\alpha}e^{\alpha x} = xe^{\alpha x}!$$

Method 3:

There is yet one more method, called the method of variation of constants which gives a second solution directly in terms of the first solution .We shall show how this method works for more general ordinary differential equations. After obtaining the final result we shall apply it to the case of the example Eq.(8) for which one solution $y_1(x) = e^{\alpha x}$ is already known. In this method one writes the second solution as

$$y(x) = u(x)y_1(x) \tag{16}$$

and demand that the differential equation satisfied by $u(x)$ be of one order lower. In this example equation for u will of order 1.

Let $y_1(x)$ be a solution of the equation

$$\left[\frac{d^2}{dx^2} + a(x)\frac{d}{dx} + b(x)\right] y(x) = 0 \tag{17}$$

substituting Eq.(16) in Eq.(17) we get

$$\frac{d^2[u(x)y_1(x)]}{dx^2} + a(x)\frac{d[u(x)y_1(x)]}{dx} + b(x)u(x)y_1(x) = 0 \tag{18}$$

Using the fact that $y_1(x)$ satisfies the original equation Eq.(17) we get an equation for $u(x)$ as

$$y_1(x)\frac{d^2u(x)}{dx^2} + a(x)y_1(x)\frac{du(x)}{dx} + 2\frac{dy_1(x)}{dx}\frac{du(x)}{dx} = 0. \tag{19}$$

Defining $v(x)$ by

$$v(x) = \frac{du(x)}{dx} \tag{20}$$

the Eq.(19) takes the form

$$y_1(x)\frac{dv(x)}{dx} + a(x)y_1(x)v(x) + 2\frac{dy_1(x)}{dx}v(x) = 0 \tag{21}$$

This equation is of first order and can be solved for $v(x)$, this solution in turn gives $u(x)$. To solve Eq.(21) we multiply it by $y_1(x)$ and rearrange in the form

$$\frac{d}{dx}\left[y_1^2(x)v(x)\right] = -a(x)y_1^2(x)v(x) \tag{22}$$

To solve the above equation, we introduce $w(x) \equiv y_1^2(x)v(x)$ and we thus get

$$\frac{1}{w(x)}\frac{dw(x)}{dx} = -a(x). \tag{23}$$

Hence we get

$$w(x) = c\exp\left[-\int^x a(t)dt\right] \tag{24}$$

where c is a constant of integration. Substituting $w(x) = y_1^2(x)v(x)$ in (24), and defintion of $v(x)$ from Eq.(20) gives the first order differential equation

$$\frac{du(x)}{dx} = \frac{c}{y_(x)^2}\exp\left[-\int^x a(t)dt\right] \tag{25}$$

which is integrated to give

$$u(x) = c\int \frac{1}{y_1^2(x)}e^{[-\int^x a(t)dt]}dx \tag{26}$$

$$y_2(x) = y_1(x)u(x) = cy_1(x)\left\{\int \frac{1}{y_1^2(x)}e^{[-\int^x a(t)dt]}dx + d\right\} \tag{27}$$

where d is another constant of integration. Eq.(8)is a special case of Eq.(17) with $a(x) = -2\alpha$, $b(x) = \alpha^2$ and the one known solution is $y_1(x) = e^{-\alpha x}$. Making these substitutions in Eq.(27) the second solution is easily computed to be

$$y(x) = cxy_1(x) + d'e^{-\alpha x}, \qquad \text{with} d' = cd. \tag{28}$$

This coincides with the a general linear combination of the two standard known solutions.

In general, $y(x)$ so obtained will be a linear combination of solutions obtained by other methods. All the three methods can be generalized to include cases of higher multiplicities and higher order differential equations.

§2 Frobenius Method of Series Solution

The Frobenius method of series solution is a useful method for a large class of linear ordinary differential equations. However many of the ordinary linear differential equations of mathematical physics are of second order and we will limit our discussion to series solution of second order, linear, ordinary differential equations only. The method of course can be generalised to linear differential equations of higher order. In this method of series solution, ordinary linear differential equations, one starts with a trial solution of the form

$$y(x, c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{29}$$

The trial solution is substituted in the differential equation $Ly = 0$ where L is a linear differential operator of the form

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x). \tag{30}$$

Next we demand that the coefficient of each power of x be zero. The resulting equations determine the index c and the coefficients a_n. At first we shall discuss the method by means of examples. Later we shall discuss a theorem which tell us the conditions under which this method will give rise to n linearly independent solutions. The relevant theorem, known as Fuch's theorem also tell us the minimum radius of convergence of the solution obtained in the series form.

The details of the method of series solution depend on the roots of the indicial equation. We will illustrate the four cases that arise by means of Bessel's equation.

Bessel's equation as an example: To introduce the method we take up the Bessel's equation as an example. The Bessel's equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0. \tag{31}$$

Substituting Eq.(29)in Eq.(31) we get

$$x^2\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + x\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + (x^2-\nu^2)\sum_{n=0}^{\infty} a_n x^{n+c} = 0. \tag{32}$$

Rewriting Eq.(32) as

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c}+\sum_{n=0}^{\infty} a_n(n+c)x^{n+c}+\sum_{n=0}^{\infty} a_n x^{n+c+2}-\nu^2\sum_{n=0}^{\infty} a_n x^{n+c}=0, \tag{33}$$

we see that the lowest power of x in the above equation is x^c. Equating the coefficient of x^c in Eq.(33)to zero we get

$$a_0c(c-1)+a_0c-\nu^2 a_0=0, \tag{34}$$

$$a_0(c^2-\nu^2)=0. \tag{35}$$

Assuming $a_0 \neq 0$ we get

$$c^2-\nu^2=0. \tag{36}$$

This equation determine the index c and is called the indicial equation. For the present case the two possible values of are c_1 and c_2 where

$$c_1=\nu,\ c_2=-\nu. \tag{37}$$

We continue and equate coefficient of the next power x^{c+1} to zero. This gives

$$a_1((c+1)^2-\nu^2)=0 \Rightarrow a_1=0 \quad (\because (c+1)^2-\nu^2\neq 0). \tag{38}$$

Equating the coefficient of x^{c+m} to zero gives recurrence relation

$$a_n=-\frac{a_{n-2}}{(n+c)^2-\nu^2}. \tag{39}$$

The recurrence relations show that higher coefficients are determined in terms of a_0 and a_1, the even terms are proportional to a_0 and the odd ones being proportional to a_1. .

For a second order linear differential equation the following four cases arise. How different cases arise is illustrated below by means of example of Bessel's equation for different values of parameter ν.

CASE-I : The roots of indicial equation are distinct and the difference of the roots is not an integer.

For Bessel's equation this is the case when $2\nu\neq$ integer. This case poses no problem and the solution for both values of c can be written down using the recurrence relations and one gets the two linearly independent solutions one for each value of the index c.

CASE-II : The roots of indicial equation are equal. For Bessel's equation this is the case when $\nu=0$ and one seems to get only one solution.

CASE-III : The difference of the roots of indicial equation is a nonzero integer and some coefficient a_n becomes infinite.

The Bessel's equation with $2\nu=$ integer comes under this case. Substituting $c=-\nu$ in the recurrence relation we get

$$a_n=\frac{a_{n-2}}{(n+c-\nu)(n+c+\nu)}=\frac{a_{n-2}}{(n-2\nu)(n)} \tag{40}$$

and one notices that the $n-2\nu$ in the denominator becomes zero when 2ν is an integer and the coefficient a_6, becomes infinite. Thus, for example for $\nu = 3$, $a_6, a_8, ...$ all become infinite due to presence of $(n-2\nu)$ factor in the denominator.

CASE-IV : The difference of the roots of indicial equation is a non-zero integer and some coefficient a_n becomes indeterminate.

To understand how this case arises, consider the Bessel's equation for 2ν = odd integer, say 5. In this case again, Eq.(40) shows that, a_5 has a zero in the denominator. However, it should be noticed that the recurrence relations imply that a_5 is proportional to a_1 which itself is zero. Thus a_5 becomes "(zero/zero)" and is therefore indeterminate. This makes all the subsequent coefficients $a_7, a_9, ...$ indeterminate.

The above discussion illustrates the four cases that may arise while applying the method of series solution to a differential equation. Each of these cases can be handled and two linearly independent solutions can be constructed when the method is applicable. Theorem by Fuchs tells us when the method can be applied and convergence properties of the series solution obtained,

§3 The Series About Point at Infinity

For a second order linear differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{41}$$

sometimes instead of a series solution in powers of x, it may be useful to expand in negative powers of x:

$$y(x,c) = x^c \sum_{n=0}^{\infty} a_n x^{-n} \tag{42}$$

This results on convergence etc. of this type of solutions are conveniently obtained by changing the independent variable from x to $t = 1/x$. The differential equation Eq.(41) written in terms of t becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \tilde{p}(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \tilde{q}(t)y = 0 \tag{43}$$

where

$$\tilde{p}(t) = \frac{2}{t} - \frac{1}{t^2}p(t); \tilde{q}(t) = \frac{1}{t^4}q(1/t) \tag{44}$$

The behaviour of the series solution at $t = 0$ gives the answer for the behaviour of the solution in the inverse powers of x.

§4 Series Solution Case-I

Case-I: Example

In this lecture we shall take up solution of an ordinary differential equation by the method of series solution. The example to be discussed is such that the indicial equation has two distinct roots and the difference of the roots is not an integer.

Consider the equation

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \tag{45}$$

Let us assume a solution in the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{46}$$

where c and a_n are to be fixed. Substituting Eq.(46) in the differential Eq.(45) we get

$$4x\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + 2\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{47}$$

or,

$$\sum_{n=0}^{\infty} 4a_n(n+c)(n+c-1)x^{n+c-1} + \sum_{n=0}^{\infty} 2a_n(n+c)x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{48}$$

or,

$$\sum_{n=0}^{\infty} 2(n+c)(2n+2c-1)a_n x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{49}$$

We now equate coefficients of different powers of x to zero. The minimum power of x in Eq.(49) is x^{c-1}. So we get

$$\text{Coeff of } x^{c-1}: \quad a_0 2c(2c-1) = 0 \tag{50}$$

$$\text{Coeff of } x^{c}: \quad a_1 2(c+1)(2c+1) + a_0 = 0 \tag{51}$$

$$\text{or,} \quad a_1 = -\frac{a_0}{2(c+1)(2c+1)} \tag{52}$$

$$\text{Coeff of } x^{c+1}: \quad a_2 2(2+c)(4+2c-1) + a_1 = 0 \tag{53}$$

$$\text{or,} \quad a_2 = -\frac{a_1}{(2c+3)(2c+4)} \tag{54}$$

$$\text{Coeff of } x^{c+m}: a_{m+1} 2(m+c+1)(2m+2c+1) + a_m = 0 \tag{55}$$

$$\text{or} \quad a_{m+1} = -a_m \frac{1}{2(m+c+1)(2m+2c+1)} \tag{56}$$

The Eq.(50) gives the indicial equation

$$2c(2c+1) = 0 \tag{57}$$

$$\text{or,} \;\; c = 0, \frac{1}{2} \tag{58}$$

Solution for c = 0

The recurrence relation Eq.(56) becomes

$$a_{m+1} = -\frac{1}{(2m+2)(2m+1)}a_m \tag{59}$$

$$\text{Therefore,} \quad a_1 = -\frac{1}{2\cdot 1}a_0 \tag{60}$$

$$a_2 = -\frac{1}{4\cdot 3}a_1 \quad = \frac{1}{4\cdot 3\cdot 2\cdot 1}a_0 \tag{61}$$

and

$$a_3 = -\frac{1}{6\cdot 5}a_2 \quad = \frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}a_0 \tag{62}$$

$$\text{Thus } a_m = \frac{(-1)^m}{(2m)!}a_0 \tag{63}$$

and one solution for, $c = 0$, is

$$y_{\mathrm{I}}(x) = x^c\sum_{n=0}^{\infty} a_n x^n = a_0\left\{1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots + \frac{(-1)^m x^m}{(2m)!} + \ldots\right\} \tag{64}$$

Solution for $c = \frac{1}{2}$

In this case we have

$$a_{m+1} = -\frac{1}{(2m+3)(2m+2)}a_m \tag{65}$$

Therefore,

$$a_1 = -\frac{1}{3\cdot 2}a_0 \tag{66}$$

$$a_2 = -\frac{1}{5\cdot 4}a_1 = \frac{1}{5\cdot 4\cdot 3\cdot 2}a_0 \tag{67}$$

$$a_3 = -\frac{1}{7\cdot 6}a_2 = -\frac{1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}a_0 \tag{68}$$

In general,

$$a_m = \frac{(-1)^m}{(2m+1)!}a_0 \tag{69}$$

The second solution is, therefore, given by

$$y_{\mathrm{II}} = x^c\sum_{n=0}^{\infty} a_n x^n = a_0 x\left\{1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots + \frac{(-1)^m x^m}{(2m+1)!} + \ldots\right\} \tag{70}$$

The most general solution of the differential equation Eq.(45) is given by

$$y(x) = \alpha y_{\mathrm{I}}(x) + \beta y_{\mathrm{II}}(x) \tag{71}$$

§5 Series Solution Case-II

In this method we shall take up solution of an ordinary differential equation by the method of series solution. In this chapter we discuss two examples for which the indicial equation has two equal roots.

Case II : Example

The first example is the differential equation $Ly = 0$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \tag{72}$$

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{73}$$

$$\frac{d}{dx}y(x,c) = \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} \tag{74}$$

$$\frac{d^2}{dx^2}y(x,c) = \sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} \tag{75}$$

Substituting Eq.(73), Eq.(74) and Eq.(75) in the differential equation Eq.(72) gives

$$x\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{76}$$

or,

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-1} + \sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{77}$$

or,

$$\sum_{n=0}^{\infty} a_n(n+c)^2 x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{78}$$

Before we start equating the coefficients of different powers of x to zero, we derive a result for later use (see Eq.(81)below).

We split off the $n = 0$ term from the remaining series in the first term in Eq.Eq.(78) and rewrite the l.h.s of Eq.(78) as

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0c^2x^{c-1} + \sum_{n=0}^{\infty} a_n(n+c)^2x^{n+c-1} + \sum_{n=0}^{\infty} a_n x^{n+c} \tag{79}$$

In the first summation in the r.h.s. we replace n with $m+1$ and sum over m from 0 to ∞; while in the second summation we simply replace $n$with$m$. This gives

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0c^2x^{c-1} + \sum_{m=0}^{\infty} a_{m+1}(m+c+1)^2x^{m+c} + \sum_{m=0}^{\infty} a_m x^{m+c} \tag{80}$$

or

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = a_0c^2x^{c-1} + \sum_{m=0}^{\infty} \left[a_{m+1}(m+c+1)^2 + a_m\right] x^{m+c} \tag{81}$$

We now equate coefficients of different powers of x to zero. The minimum power of x in Eq.(78) is x^{c-1}. So we get

$$\text{Coefficient of } x^{c-1}\text{: } a_0c^2 = 0 \tag{82}$$

Assuming $a_0 \neq 0$ we get the indicial equation $c^2 = 0$. Thus the roots of the indicial equation are coincident and we have

$$c = 0. \tag{83}$$

$$\text{Coeff of } x^c \text{ in } Eq.(81) : a_1(c+1)^2 + a_0 = 0 \tag{84}$$

$$a_1 = -\frac{a_0}{(c+1)^2} \tag{85}$$

$$\text{Coeff of } x^{c+1} \text{ in } Eq.(81) : a_2(2+c)^2 + a_1 = 0 \tag{86}$$

$$\text{or, } a_2 = -\frac{a_0}{(c+2)^2(c+1)^2} \tag{87}$$

$$\text{Coeff of } x^{c+m} \text{ in } Eq.(81) : a_{m+1} = -\frac{a_m}{(m+c+1)^2} \tag{88}$$

This gives

$$a_m = (-1)^m \ \frac{a_0}{[(c+m)(c+m-1)\dots(c+1)]^2} \tag{89}$$

and hence

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} = a_0 \sum_{n=0}^{\infty} (-1)^n \ \frac{x^{n+c}}{[(c+n)(c+n-1)\dots(c+1)]^2} \tag{90}$$

Notice that, if we use Eq.(89) in Eq.(81), one gets that for $c \neq 0$ $y(x,c)$ satisfies the relation

$$x\frac{d^2y(x,c)}{dx^2} + \frac{dy(x,c)}{dx} + y(x,c) = a_0 c^2 x^{c-1} \tag{91}$$

The right hand side of the above equation vanishes if we set $c = 0$, showing that $y(x,c)|_{c=0}$ is a solution of the differential equation.

Also if we differentiate Eq.(91) w.r.t. c and set $c = 0$, the right hand side again vanishes showing that $\left.\frac{dy(x,c)}{dc}\right|_{c=0}$ is also a solution. Thus two linearly independent solutions are given by

$$y_{\mathrm{I}}(x,c) = y(x,c)|_{c=0} \text{ and } y_{\mathrm{II}}(x,c) = \left.\frac{dy(x,c)}{dx}\right|_{c=0} \tag{92}$$

We shall now determine the series for the two solutions in Eq.(92). The coefficients a_m can be expressed in terms of gamma functions $\Gamma(x)$ making use of the property

$$\Gamma(z+1) = z\Gamma(z) \tag{93}$$

Using Eq.(93) repeatedly, for $r < n$ we get

$$\begin{aligned} \Gamma(z+n+1) &= (z+n)\Gamma(z+n) \\ &= (z+n)(z+n-1)\Gamma(z+n-1) \\ &= \dots\dots \\ &= (z+n)(z+n-1)(z+n-2)\dots(z+r)\Gamma(\\ \text{Or,} \quad (z+n)(z+n-1)\dots(z+r) &= \frac{\Gamma(z+n+1)}{\Gamma(z+r)} \end{aligned}$$

On using Eq.(96) with $z = c$, $r = 1$we get

$$(c+n)(c+n-1)\ldots\ldots(c+1) = \frac{\Gamma(c+n+1)}{\Gamma(c+1)} \tag{97}$$

Use Eq.(97) to rewrite Eq.(90) to get

$$y(x,c) = a_0[\Gamma(c+1)]^2 \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+c}}{[\Gamma(c+n+1)]^2} \tag{98}$$

Setting $a_0[\Gamma(c+1)]^2 = 1$, we get

$$y(x,c) = x^c \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{[\Gamma(c+n+1)]^2} \tag{99}$$

The two solutions of the given differential equation are

$$y_1(x) = y(x,c)|_{c=0} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{[\Gamma(n+1)]^2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(n!)^2} \tag{100}$$

and

$$y_2(x,c) = \left.\frac{dy(x,c)}{dc}\right|_{c+0} \tag{101}$$

For the second solution the derivatives w.r.t. c, at $c = 0$, are needed and can be conveniently expressed in terms of the $\Gamma(x)$ and the function $\psi(x)$, where

$$\psi(x) = \frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx} = \frac{d}{dx}\log\Gamma(x) \tag{102}$$

Therefore, computing the derivative of $\frac{1}{[\Gamma(x)]^2}$

$$\frac{d}{dx}\frac{1}{[\Gamma(x)]^2} = -2\frac{1}{[\Gamma(x)]^3}\frac{d\Gamma(x)}{dx} = -2\frac{\psi(x)}{[\Gamma(x)]^2} \tag{103}$$

Differentiating $y(x,c)$ given by Eq.(99) we get

$$\frac{dy(x,c)}{dc} = x^c \ \log x \sum_{n=0}^{\infty}(-1)^n \ \frac{x^n}{[\Gamma(c+n+1)]^2} \ + \ x^c \sum_{n=0}^{\infty}(-1)^n \frac{d}{dc}\frac{1}{[\Gamma(c+n-} \tag{104}$$

Hence

$$y_2(x) = \left.\frac{dy(x,c)}{dc}\right|_{c=0} = y_1(x)\log x - 2\sum_{n=0}^{\infty}(-1)^n \ x^n \ \frac{\psi(n+1)}{(n!)^2} \tag{105}$$

where in the last step in Eq.(105) we have used Eq.(103) to get

$$\left.\frac{d}{dc}\frac{1}{[\Gamma(c+n+1)]^2}\right|_{c=0} = -2 \ \frac{\psi(n+1)}{[\Gamma(n+1)]^2} = -2\frac{\psi(n+1)}{(n!)^2} \tag{106}$$

The most general solution is a linear combination of $y_1(x)$ and $y_2(x)$

$$y(x) = \alpha \ y_1(x) \ + \ \beta \ y_2(x) \tag{107}$$

Question: In going from Eq.(98)to Eq.(99) we have made a choice

$$a_0 = \frac{1}{[\Gamma(c+1)]^2}$$

How will the solution y_2 change if we had proceed without making this choice ? It can be verified that the most general form Eq.(107) of the solution is not affected by this choice

§6 Series Solution Case-III

We shall now take up the series solution for differential equations when the roots of the indicial equation differ by an integer $\neq 0$. For such equations two different possibilities arise.The first possibility, discussed in this lecture is that roots of the indicial equation differ by an integer and this results in some coefficient becoming infinite. In the other possibility, to be taken up in the next lecture, is when some coefficient becomes indeterminate.

Case-III : Example

An example of the case-III is the ordinary differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (2x+1)y = 0 \tag{108}$$

which we write as $Ly = 0$ where

$$Ly = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (2x+1)y. \tag{109}$$

Let

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{110}$$

Then we have

$$\frac{d}{dx}y(x,c) = \sum_{n=0}^{\infty} a_n (n+c)x^{n+c-1} \tag{111}$$

$$\frac{d^2}{dx^2}y(x,c) = \sum_{n=0}^{\infty} a_n (n+c)(n+c-1)x^{n+c-2} \tag{112}$$

Substituting in the given differential equation Eq.(108), we get

$$x^2\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + x\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} - (2x+1)\sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{113}$$

Or we have,

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c} + x\sum_{n=0}^{\infty} a_n(n+c)x^{n+c-1} - (2x+1)\sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{114}$$

This can be rearranged as

$$\sum_{n=0}^{\infty} a_n\left\{(n+c)^2 - 1\right\}x^{n+c} - 2\sum_{n=0}^{\infty} a_n x^{n+c+1} = 0 \tag{115}$$

Before we start equating the coefficients of different powers of x to zero, we derive a result for later use (see Eq.(118) below) for later use. We split off the $n=0$ term first sum and write it separately.

$$a_0(c^2-1)x^c+\sum_{n=0}^{\infty} a_n\left\{(n+c)^2-1\right\}x^{n+c}-2\sum_{n=0}^{\infty} a_n x^{n+c+1}=0 \quad (116)$$

The summation index in the second sum can be redefined from n to $r = n+1$, so that sum over r runs from 0 to ∞. Thus we get

$$Ly(x,c)=a_0(c^2-1)x^c+\sum_{r=0}^{\infty} a_{r+1}\left\{(r+1+c)^2-1\right\}x^{r+1+c}-2\sum_{r=0}^{\infty} a_r x^{r+c+1}=0 \quad (117)$$

The left hand side of Eq.(115) is just $Ly(x,c)$. Eq.(117) enables us to rewrite $Ly(x,c)$ as

$$Ly(x,c)=a_0(c^2-1)x^c+\sum_{r=0}^{\infty} a_{r+1}\left\{[(r+1+c)^2-1]-2a_r\right\}x^{r+1+c}=0 \quad (118)$$

Eq.(118) will be needed below, for the moment we get back to Eq.(115). The minimum power of x in Eq.(118) is x^c to zero we get,

$$a_0(c^2-1)=0 \quad (119)$$

The indicial equation is, therefore, given by $c^2-1=0$ and the possible values of c are ± 1. Equating the coefficients of successive powers x^{c+1}, x^{c+2} etc. to zero gives

$$\text{Coefficient of } x^{c+1}: \quad a_1\left[(c+1)^2-1\right]-2a_0=0 \quad (120)$$

$$\text{therefore} \quad a_1=\frac{2a_0}{c(c+2)} \quad (121)$$

$$\text{Coefficient of } x^{c+2}: a_2[(c+2)^2-1]-2a_1=0 \quad (122)$$

and hence

$$a_2=\frac{2a_1}{(c+1)(c+3)}=\frac{2.2a_0}{(c+1)(c+3)(c+2)c} \quad (123)$$

Note that the coefficient a_2 becomes infinite when $c=-1$. Similarly, the coefficient of x^{c+3} equated to zero implies

$$\text{therefore} \quad a_3=\frac{2a_2}{(c+4)(c+2)} \quad (124)$$

The recurrence relation as obtained from Eq.(115) by demanding that the coefficient of x^{m+c} be zero.

$$(m+c+1)(m+c-1)a_m-2a_{m-1}=0 \quad (125)$$

$$a_m=\frac{2a_{m-1}}{(m+c+1)(m+c-1)} \quad (126)$$

Thus a_3, and all the subsequent coefficients, are proportional to a_2 and hence becomes infinite for $c=-1$, due to presence of a factor $(c+1)$ in the denominator of a_2, see Eq.(123). Since the overall constant a_0 arbitrary, we may select $a_0 = k(c+1)$ making a_3 and all the subsequent coefficients finite for both the values of $c = \pm 1$. With the choice $a_0 = k(c+1)$ and using the recurrence relation Eq.(126) in Eq.(118) one gets

$$Ly(x,c) = a_0(c^2-1)x^c = k(c-1)(c+1)^2 x^c \tag{127}$$

It is apparent from the above equation that for $c = 1$ we have two linearly independent solutions given by

$$y_1(x) = y(x,c)\big|_{c=-1} \text{ and } y_2(x) = \left.\frac{dy(x,c)}{dc}\right|_{c=-1} \tag{128}$$

It can be explicitly checked that yet another solution, obtained from $y(x,c)$ by setting $c = -1$, is proportional to the solution $y_1(x)$.

We shall now get explicit form of the two solutions Eq.(128), Eq.(121)Eq.(123),Eq.(124) and Eq.(126) give

$$a_1 = \frac{2a_0}{(c)(c+2)}, \tag{129}$$

$$a_2 = \frac{2.2a_0}{(c+3)(c+2)(c+1)c}, \tag{130}$$

$$a_3 = \frac{2^3 a_0}{(c+4)(c+3)(c+2)(c+2)(c+1)c}. \tag{131}$$

In general

$$a_m = \frac{2^m a_0}{(m+c+1)(m+c)\cdots(c+2)(m+c-1)(m+c-2)\cdots c} \tag{132}$$

Multiplying and dividing Eq.(132) by $[\Gamma(c+2)]^2$, the expression for a_m can be easily cast in the form

$$a_m = \frac{2^m a_0 \Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c(c+1)} \tag{133}$$

Writing the series for $y(x,c)$, using Eq.(130),Eq.(131) and Eq.(133), we obtain

$$y(x,c) = a_0 x^c \left\{ 1 + \frac{2x}{(c)(c+2)} + \frac{2^2\,x^2}{(c+1)c(c+3)(c+2)} + \cdots + \frac{2^m\Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c(c+1)} x^m + \cdots \right\} \tag{134}$$

Next we use $a_0 = k(c+1)$ and rewrite the above series as

$$y(x,c) = kx^c \left\{ (c+1) + (c+1) + \frac{2x}{(c)(c+2)} + \frac{2^2\,x^2}{c(c+3)(c+2)} + \cdots + \frac{2^m\Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c} x^m + \cdots \right\} \tag{135}$$

One solution is obtained by setting $c = -1$ in Eq.(135). Apart from an overall constant, the first solution can be written as

$$y_1(x) = \sum_{m=2}^{\infty} \frac{2^m}{(m!)(m-2)!} x^{m-1} = 2\sum_{m=0}^{\infty} \frac{2^{m+1}}{(m!)(m+2)!} x^{m+1} \tag{136}$$

To obtain the second solution we differentiate Eq.(135) w.r.t c and set $c = -1$. This gives

$$
\begin{aligned}
\frac{dy(x,c)}{dc} &= k\log x\, x^c \left\{(c+1) + (c+1)\frac{2x}{c(c+2)} + \frac{2^2\, x^2}{c(c+3)(c+2)} + \cdots \right.\\
&\qquad \left. \cdots + \frac{2^m\Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c)} x^m + \cdots \right\}\\
&\quad + x^c\left\{1 - \frac{2x}{c^2} + \frac{\mathrm{d}}{\mathrm{d}c}\frac{2^2\, x^2}{c(c+3)(c+2)} + \cdots + \frac{\mathrm{d}}{\mathrm{d}c}\frac{2^m\Gamma(c+2)\Gamma(c+2)}{\Gamma(m+c+2)\Gamma(m+c)c)}x^m + \cdots\right\}
\end{aligned} \tag{137}
$$

Computing the derivative of $\log a_m$, with a_m as in Eq.(133) we get

$$\log a_m = \log 2^m k + 2\log\Gamma(c+2) - \log\Gamma(m+c+2) - \log\Gamma(m+c) - \log c \tag{138}$$

Thus we have and setting $c = -1$ one gets

$$
\begin{aligned}
\frac{1}{a_m}\left.\frac{da_m}{dc}\right|_{c=-1} &= \left.\left\{2\psi(c+2) - \psi(c+m+2) - \psi(m+c) - \frac{1}{c}\right\}\right|_{c=-1}\\
&= -2\gamma - \psi(m+1) - \psi(m-1) + 1
\end{aligned} \tag{139}
$$

Writing

$$\phi(n) = 1 + \frac{1}{2} + \frac{1}{3} = \cdots + \frac{1}{n} \tag{140}$$

and defining $\phi(0) = 0$, and using

$$\psi(n) = -\gamma + \phi(n-1), \text{for n ¿ 1} \tag{141}$$

Eq.(139) can be simplified, for $m > 2$ to give

$$\left.\frac{da_m}{dc}\right|_{c=-1} = a_m\left[\phi(m-2) + \phi(m) - 1\right] \tag{142}$$

$$= \frac{2^m\left[\phi(m-2) + \phi(m) - 1\right]}{m!\,(m-2)!} \tag{143}$$

Substituting back in Eq.(137) gives the series for the second solution as

$$y_2(x) = -2y_1(x)\log x + x^{-1}\Delta(x) \tag{144}$$

where $\Delta(x)$ is the series given by

$$
\begin{aligned}
\Delta(x) &= 1 - 2x + x^2 + \cdots + \frac{2^m\left[\phi(m) + \phi(m-2) - 1\right]}{m!\,(m-2)!}x^m + \cdots\\
&= \left\{1 - 2x + x^2 + \cdots + \sum_{m=3}^{\infty}\frac{2^m\left[\phi(m) + \phi(m-2) - 1\right]}{m!\,(m-2)!}x^{m+1} + \cdots\right\}
\end{aligned}
$$

The series solution obtained by setting $c = 1$ in $y(x,c)$ of Eq.(135)is proportional to $y_1(x)$.

§7 Series Solution Case-IV

In this lecture we shall take up solution of an ordinary differential equation by the method of series solution. The example to be discussed is such that the difference of the roots of the indicial equation is an integer and some coefficient becomes indeterminate.

Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x^2y = 0 \tag{145}$$

Substituting

$$y = \sum_{n=0}^{\infty} a_n x^{n+c} \tag{146}$$

in Eq.(145) we get

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + x^2\sum_{n=0}^{\infty} a_n x^{n+c} = 0 \tag{147}$$

or,

$$\sum_{n=0}^{\infty} a_n(n+c)(n+c-1)x^{n+c-2} + \sum_{n=0}^{\infty} a_n x^{n+c+2} = 0 \tag{148}$$

The lowest power of x in the right hand side of Eq.(148) is x^{c-2}. This gives

$$a_0c(c-1) = 0 \tag{149}$$

Therefore the two values of c are $c = 0$ and $c = 1$. Equating the coefficients of $x^{c-1}, x^c, x^{c+1}, x^{c+2}, \ldots$ to zero successively gives

$$a_1c(c+1) = 0, \tag{150}$$

$$a_2(c+1)(c+2) = 0, \tag{151}$$

$$a_3(c+2)(c+3) = 0, \tag{152}$$

$$a_4(c+4)(c+3) + a_0 = 0. \tag{153}$$

The recurrence relation obtained by considering the coefficient of x^{m+c+2} is

$$a_{m+4}(c+m+4)(c+m+3) + a_m = 0. \tag{154}$$

The solution for $c = 1$ can be constructed easily using the recurrence relations.

Let us now look at the case $c = 0$. In this case, fromEq.(150) we get

$$a_1.0 = 0. \tag{155}$$

Thus a_1 cannot be fixed and is indeterminate. In this case we proceed as before except that we retain both a_0 and a_1 as unknown parameters. We construct solution for this case, $c = 0$, first and then come back and look at the solution for $c = 1$.

Case $c = 0$:

Substituting $c = 0$ from Eq.(150) to Eq.(154) we get

$$a_2 = a_3 = 0; \quad a_4 = -\frac{a_0}{4.3} \tag{156}$$

$$a_{m+4} = -\frac{a_m}{(m+4)(m+3)} \tag{157}$$

Combining Eq.(156) and Eq.(157) we see that

$$a_2 = a_6 = a_1 0 \cdots 0 \tag{158}$$

and

$$a_3 = a_7 = a_1 1 \cdots 0 \tag{159}$$

Also

$$a_4 = -\frac{1}{4.3}a_0; \quad a_8 = -\frac{1}{8.7}a_4; \quad a_{12} = -\frac{1}{12.11}a_8 \tag{160}$$

$$a_5 = -\frac{1}{5.4}a_1; \quad a_9 = -\frac{1}{9.8}a_5; \quad a_{13} = -\frac{1}{13.12}a_9 \tag{161}$$

Solving Eq.(160) and Eq.(161) we get

$$a_4 = -\frac{1}{4.3}a_0; \quad a_8 = -\frac{1}{8.7.4.3}a_0; \quad a_{12} = -\frac{1}{12.11.8.7.4.3}a_0 \tag{162}$$

$$a_5 = -\frac{1}{5.4}a_1; \quad a_9 = -\frac{1}{9.8.5.4}a_1; \quad a_{13} = -\frac{1}{13.12.9.8.5.4}a_1 \tag{163}$$

The series solution in this case contains two parameters, which are not determined by the recurrence relations, and is given by

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \tag{164}$$

$$y_1(x) = 1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \frac{x^{12}}{3.4.7.8.11.12} + \cdots \tag{165}$$

$$y_2(x) = x\left\{1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \frac{x^{12}}{4.5.8.9.12.13} + \cdots\right\} \tag{166}$$

These two functions $y_1(x)$ and $y_2(x)$ represent two linearly independent solutions. What happens when one tries to construct the solution for the second value of c ? In this case we recover one of the above two solutions already obtained. This will now be demonstrated explicitly.

Case $c = 1$ **:**

In this case we get

$$a_1 = a_2 = a_3 = 0 \tag{167}$$

$$a_{m+4} = -\frac{a_m}{(m+5)(m+4)} \tag{168}$$

We therefore get

$$a_4 = -\frac{1}{5.4}a_0; \quad a_8 = -\frac{1}{9.8}a_4; \quad a_{12} = -\frac{1}{13.12}a_8 \tag{169}$$

Compare the equations Eq.(169) with Eq.(161). We now construct the series

$$y = x^c \sum a_n x^n \tag{170}$$

and get

$$y_2(x) = a_0 x\left\{1 - \frac{x^4}{4.5} + \frac{x^8}{4.5.8.9} - \frac{x^{12}}{4.5.8.9.12.13} + \cdots\right\} \tag{171}$$

This solution coincides with $y_2(x)$ of Eq.(166) except for an overall constant. Hence the most general solution of the differential equation Eq.(145) is given by Eq.(164).

§8 A Summary of Method of Series Solution

Summary of the method of series solution

The Frobenius method of series solution is a useful method for a large class of linear ordinary differential equations of mathematical physics. A general ordinary second order linear differential equation can be put in the form

$$\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y(x)}{\mathrm{d}x} + q(x)y(x) = 0 \tag{172}$$

In the Frobenius method of series solution, it is assumed that the solution can be written in the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n x^{n+c} = x^c\left[a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots\right] \tag{173}$$

The parameter c is called ***index***. The expansion parameters a_n and the index c are determined by substituting Eq.(173) in the ODE Eq.(172), expanding $p(x)$ and $q(x)$ in powers of x, and comparing the coefficients of different powers of x on both sides. The coefficient of the general power n equated to zero gives recurrence relations for the coefficients of expansion a_n. These recurrence relations are then solved and the expansion coefficients are fixed.

When this method is applicable, one gets two linearly independent solutions $y_1(x)$ and $y_2(x)$ for the second order differential equations. The most general solution $y(x)$ of the ODE Eq.(172) is then represented as a linearly combination of the solutions $y_1(x)$ and $y_2(x)$.

$$y_1(x) = \alpha y_1(x) + \beta y_2(x) \tag{174}$$

where the constants α and β are to be fixed by initial conditions. It must be remarked that the two linear independent solutions are not always of the form Eq.(173) assumed in the beginning. In general one may also get a series of type Eq.(173) multiplied by $\log x$. The expansion Eq.(173) is expansion about the point $x = 0$. In general, one may attempt a series solution about any point x_0. In such a case, instead of Eq.(173), one assumes the solution to be of the form

$$y(x,c) = \sum_{n=0}^{\infty} a_n(x - x_0)^c + n \tag{175}$$

We now summarize the method of obtaining two linearly independent solutions in the four cases of series solution.

Case I: In this case the roots of the indicial equation are distinct and the difference of the roots c_1 and c_2 is not an integer. The two linearly independent solutions are given by

$$y_1(x) = y(x,c)|_{c=c_1} \text{ and } y_2(x) = y(x,c)|_{c=c_2}$$

Case II: In this case the roots of the indicial equation are equal to, say, c_0. The two linearly independent solutions are given by

$$y_1(x) = y(x,c)|_{c=c_0} \text{ and } y_2(x) = \left.\frac{\mathrm{d}}{\mathrm{dc}} y(x,c)\right|_{c=c_0}$$

Case III: In this case the roots of the indicial equation, c_1 and c_2 is an integer.And one of the coefficients becomes infinite for one of the values of c, which we assume to be c_1. In this case we assume $a_0 = k(c - c_1), \quad k \neq 0$. The two linearly independent solutions are then given by

$$y_1(x) = y(x,c)|_{c=c_1} \text{ and } y_2(x) = \left.\frac{\mathrm{d}}{\mathrm{dc}} y(x,c)\right|_{c=c_1}$$

The solution obtained from $y(x,c)$ by setting $c = c_2$ is identical with $y_1(x)$ apart from an over all constant.

Case IV: In this case the roots of the indicial equation, c_1 and c_2 are distinct and the difference of the roots c_1 and c_2 is an integer. And one of the coefficients, say a_{n}, becomes indeterminate for one of the values of c, which we assume to be c_1. In this case we keep a_0 and a_{n} as unknown constants, and the most general solution containing two unknown constants is obtained from $y(x,c)$ setting $c = c_1$.

$$y(x) = y(x,c)|_{c=c_1}$$

The solution obtained from $y(x,c)$ by setting $c = c_2$ coincides with $y(x)$ for particular values of the constants a_0 and a_{n}.

Convergence of Series Solutions

We are now interested in knowing the properties of the solutions.
Having obtained the solutions in a series form one must ask what are the values of x for which the series appearing in the solutions converge? When do we have two linearly independent solutions? The answer to these and related questions is given by Fuch's Theorem.For this purpose it turns out to be useful to regard the independent variable x as complex variable and to continue the two functions $p(x)$ and $q(x)$ to the complex plane.
As a preparation to the statement of the Fuch's Theorem we define an ordinary point, the regular singular and the irregular singular

points of an ordinary differential equation. As already mentioned the independent variable x will regarded as a complex variable.

A point $x = x_0$ in the complex plane is called an ***ordinary point*** of the second order linear differential equation if both the functions $p(x)$ and $q(x)$ are analytic at $x = x_0$.

A point x_0 is called ***singular point*** of the second order ordinary linear differential equation, if it is not an ordinary point.

A singular point x_0 is called ***regular singular point*** of the differential equation if the two functions $P(x)$ and $Q(x)$, where

$$P(x) = (x - x_0)p(x), \qquad Q(x) = (x - x_0)^2 q(x) \tag{176}$$

are analytic at x_0.
A singular point x_0 is called ***irregular singular point*** if it is a singular point but not a regular singular point.

Point at Infinity: The above definitions are easily extended to the point at infinity. We say that the point at infinity ($x = \infty$) is, respectively, an ordinary point, a regular singular point of a differential equation Eq.(172) if for the corresponding equation Eq.(176),in $t = 1/x$, the point $t = 0$ is an ordinary point or a regular singular point. A similar statement holds for the irregular singular points.

Theorem 1 *If x_0 is an ordinary point of the differential equation Eq.(172), there exist two linearly independent solutions which are analytic at x_0. These solutions are therefore expressible as power series in $(x - x_0)$ in the form Eq.(175). The radius of convergence of the power series is at least as large as the distance of x_0 from the nearest singular point of the functions $p(x)$ and $q(x)$ in the complex plane.*

The Fuch's theorem given below summarises the corresponding results for the series solution about a regular singular point.

Theorem 2 (Fuch's Theorem) *If the differential equation Eq.(172) has a regular singular singular point at $x = x_0$ there exist two linearly independent solutions which can be expressed in the form*

$$y(x) = (x - x_0)^c \{\log(x - x_0)\phi_1(x) + \phi_2(x)\} \tag{177}$$

where $\phi_1(x)$ and $\phi_2(x)$ have power series expansions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{178}$$

The series expansions for $\phi_1(x)$ and $\phi_2(x)$ have radius of convergence at least as large as the distance of x_0 from the nearest point, in the complex plane, of $P(x)$ and $Q(x)$ as defined in Eq.(176).