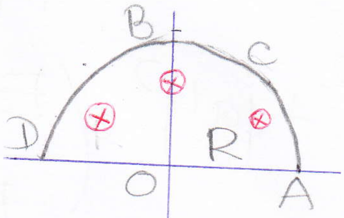
§§7.3 Q[7]

To evaluate $\int_0^\infty \left(\frac{x^2\,dx}{x^6+1}\right)$ use contour C_R^+ in the upper half plane

$$\int_0^\infty \frac{x^2\,dx}{(x^6+1)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^6+1)} .$$

$$= \lim_{R\to\infty} \frac{1}{2}\oint_{C_R^+} \frac{z^2\,dz}{(z^6+1)} \equiv I$$



only residues in the upper half plane need to be computed

$C_R^+ \to OACBDO$

The singular points are at the sixth roots of -1, denote these by ξ_i,

$$\xi_i = \exp(i\pi/6) \times \exp(2i\,n\pi/6) \qquad n = 0,1,2,3,4,5$$

Of these only the first three are in the upper half plane. These are $e^{i\pi/6}$, $e^{i\pi/2}$, $e^{i5\pi/6}$ $\quad (z^6+1=0 \Rightarrow z^6=-1 \Rightarrow z=(-1)^{1/6})$

Each ξ_i is a simple pole

$$\operatorname{Res}\left\{\frac{z^2}{(z^6+1)}\right\}_{z=\xi_i} = \lim_{z\to\xi_i} \frac{(z-\xi_i)\,z^2}{(z^6+1)}$$

$$= \xi_i^2 \lim \frac{z-\xi_i}{(z^6+1)} = \xi_i^2 \frac{1}{6\xi_i^5}$$

$$= \frac{1}{6}\frac{1}{\xi_i^3}$$

$\therefore$ Sum of residues at poles in the upper half plane

$$= \frac{1}{6}\left(e^{-i\pi/2} + e^{-3i\pi/2} + e^{-i5\pi/2}\right)$$

$$= \frac{1}{6}(-i+i-i) = -\frac{i}{6}$$

$$\therefore\ I = \frac{1}{2}\times(2\pi i)\times\left(-\frac{i}{6}\right) = \frac{\pi}{6}.$$

$$\boxed{\begin{aligned} 5\pi/2 &= 2\pi + \pi/2 \\ e^{-i5\pi/2} &= e^{-i\pi/2} \\ &= -i \end{aligned}}$$