Compute the integral

§§ 7.3

Q 3

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1}$$

using contour integration.

Solution: We first write the integral as integral along interval $(-R, R)$ of real axis in the complex plane.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^4+x^2+1)} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dz}{(z^4+z^2+1)}$$

$$= \lim_{R\to\infty} \int_{AOB} \frac{dz}{(z^4+z^2+1)}$$

A semicircular contour BCA can be added without changing the value in the limit $R \to \infty$.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^4+x^2+1)} = \lim_{R\to\infty} \int_{AOB} + \int_{BCA} \frac{dz}{(z^4+z^2+1)}$$

$$= \oint_{\Gamma} \frac{dz}{(z^4+z^2+1)}$$

Note $(z^2-1)(z^4+z^2+1) = (z^6-1)$

Therefore $z^4+z^2+1=0$ has solutions

$$\xi_k = e^{2\pi i/6 \times k}, \quad k = 1, 2, 4, 5$$

Obtained by taking 6th roots of unity except $z = \pm 1$. Of these $k = 1, 2$ lie in upper half plane. Thus We need to compute residue at

$$\xi_1 = e^{2\pi i/6} = \frac{1+i\sqrt{3}}{2} = e^{i\pi/3} \quad \text{and} \quad \xi_2 = e^{4\pi i/6} = \left(\frac{-1+i\sqrt{3}}{2}\right) = e^{2\pi i/3}$$

Residue at $z = \xi_1$

$\rightarrow \lim_{z \to \xi_1} \dfrac{z-\xi_1}{z^4+z^2+1}$

Use l'Hopital's Rule

$= \dfrac{1}{4\xi_1^3 + 2\xi_1}$

$\xi_1 = e^{2\pi i/3}$ $\quad \xi_1^3 = -1$

$\xi_1 = \dfrac{1+i\sqrt{3}}{2}$

$= \dfrac{1}{-4+(1+i\sqrt{3})} = \dfrac{1}{(i\sqrt{3}-3)}$

Residue at $z = \xi_2$

$\rightarrow \lim_{z \to \xi_2} \dfrac{z-\xi_2}{z^4+z^2+1}$

$= \dfrac{1}{4\xi_2^2 + 2\xi_2}$

$\xi_2 = e^{4\pi i/6} = \dfrac{-1+i\sqrt{3}}{2}$

$\xi_2^3 = 1$

$= \dfrac{1}{+4+2\left(\dfrac{-1+i\sqrt{3}}{2}\right)} = \dfrac{1}{(i\sqrt{3}+3)}$

$\therefore$ Sum of residues

$= \dfrac{1}{(i\sqrt{3}-3)} + \dfrac{1}{(i\sqrt{3}+3)} = \dfrac{2i\sqrt{3}}{-3-9} = \dfrac{-2i\sqrt{3}}{12}$

$= \dfrac{-i}{2\sqrt{3}}$

$\therefore I = 2\pi i \times \left(\dfrac{-i}{2\sqrt{3}}\right) = \dfrac{\pi}{\sqrt{3}}$

$\boxed{\text{April 23, 2017}}$