Q[6] §§7.3

§§7.3 Q6.

$$\int_0^\infty \frac{x-2}{x^4+10x^2+9}\,dx$$

$$= \int_0^\infty \frac{x\,dx}{x^4+10x^2+9} - 2\int_0^\infty \frac{dx}{(x^4+10x^2+9)}$$

$$\equiv I_1 + I_2 \qquad (1)$$

$$I_1 = \int_0^\infty \frac{x\,dx}{x^4+10x^2+9} \qquad (2)$$

$$= \left(\frac{-1}{2\pi i}\right) \int \frac{z \log z \; dz}{(z^4+10z^2+9)}$$

Follow Q 1 of Tutorial §§7.2

$$= \left(\frac{-1}{2\pi i}\right) \oint_C \frac{z \log z\, dz}{(z^4+10z^2+9)} \qquad (3)$$

Where C is two circles contour of Fig 7.3 in §§7.2

The singular points of the integrand in $I_1$ are poles at $z=\pm i$, $z=\pm 3i$. All the poles are included inside the two circles contour. For computation of residues let us write down the definition of $\log z$ that led to the integral in (3). The function $\log z$ has branch cut along ~~o~~ positive real axis, $\theta=0$, and

$$\log z = \log r + i\theta \qquad 0 < \theta < 2\pi$$

Note that $(z^4+10z^2+9)$ factorizes as

$$(z^4+10z^2+9) = (z^2+9)(z^2+1)$$

$$= (z+3i)(z-3i)(z+i)(z-i)$$

All poles are simple poles therefore compute residues using (6.11).

The second integral $I_2 = -2\int_0^{\infty} \frac{dx}{(x^4+10x^2+9)}$

can be evaluated by closing the contour in the upper half plane

$$I_2 = -\int_{-\infty}^{\infty} \frac{dx}{(x^4+10x^2+9)} = -\oint_{C_R^+} \frac{dz}{(z^4+10z^2+9)}$$

Only poles at $z = 3i, i$ contribute to the integral

$$\text{Residue}\left\{\frac{1}{z^4+10z^2+9}\right\}_{z=3i} = \frac{1}{(z+3i)(z^2+1)}\bigg|_{z=3i}$$

$$= \frac{1}{(6i)(-8)} = \frac{i}{48}$$

$$\text{Residue}\left\{\frac{1}{z^4+10z^2+9}\right\}_{z=i} = \frac{1}{(z^2+9)(z+i)}\bigg|_{z=i}$$

$$= \frac{1}{8(2i)} = \frac{-i}{16}$$

$$\therefore I_2 = -(2\pi i)\left(\frac{i}{48} - \frac{i}{16}\right) = 2\pi\left(\frac{1-3}{48}\right)$$

$$= -\pi/12$$

The final answer is $\left(\frac{1}{8}\ln 3 - \frac{\pi}{12}\right)$

$$= \frac{1}{24}(3\ln 3 - 2\pi) = \frac{1}{24}(\ln 27 - 2\pi)$$