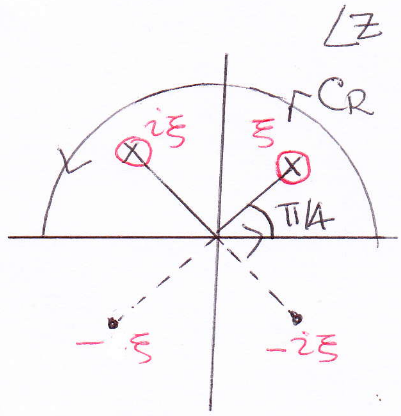
§§7.3
Q9

$$\int_0^\infty \frac{x^2+1}{(x^4+1)^2}\,dx = \frac{\pi}{2\sqrt{2}}$$

$$\int_0^\infty \frac{x^2+1}{(x^4+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2+1}{(x^4+1)^2}\,dx$$

$$= \lim_{R\to\infty} \frac{1}{2}\int_{-R}^{R} \frac{z^2+1}{(z^4+1)^2}\,dz$$



The contour can be closed by adding a semicircle of radius R in upper half plane. The semicircle does not contribute to the integral in the limit $R\to\infty$.

$$\int_0^\infty \frac{x^2+1}{(x^4+1)^2}\,dx = \frac{1}{2}\lim \oint_{C_R} \frac{z^2+1}{(z^2+1)^4}\,dz$$

Singular points in the upper half plane are at $z_{1,2} = e^{i\pi/4}, e^{3i\pi/4}$ and are double poles.

$$\text{Residue } \left.\frac{z^2+1}{(z^4+1)^2}\right|_{z=z_K} = \lim \frac{d}{dz}\frac{(z^2+1)(z-z_K)^2}{(z^4+1)^2}$$

Use ξ to denote any one of the singular points z_K.
Then $\xi^4 = -1$, $(z^4+1) = (z^4-\xi^4) = (z-\xi)(z+\xi)(z^2+\xi^2)$

$$\text{Res } \left.\frac{z^2+1}{(z^4+1)^2}\right|_{z=\xi} = \lim \frac{d}{dz}\frac{(z-\xi)^2(z^2+1)}{(z^4-\xi^4)^2}$$

$$= \lim_{z\to\xi} \frac{d}{dz}\frac{(z^2+1)}{(z+\xi)^2(z^2+\xi^2)^2}$$

Residue $\dfrac{z^2+1}{(z^4+1)^2}\Big|_{z=\xi}$ $\qquad \xi = e^{i\pi/4},\ e^{3i\pi/4}$

$\xi^4 = -1$

$$= \frac{d}{dz}\, \frac{(z-\xi)^2(z^2+1)}{(z-\xi)^2(z+\xi)^2(z^2+\xi^2)^2}\Bigg|_{z=\xi}$$

$\because\ \dfrac{1}{(z^4+1)^2} = \dfrac{1}{(z^4-\xi^4)^2} = \dfrac{1}{(z-\xi)^2(z+\xi)^2(z^2+\xi^2)^2}$

$$= \frac{d}{dz}\, \frac{(z^2+1)}{(z+\xi)^2(z^2+\xi^2)^2}\Bigg|_{z=\xi}$$

$$= \left[\frac{2z}{(z+\xi)^2(z^2+\xi^2)^2} + \frac{(z^2+1)(-2)}{(z+\xi)^3(z^2+\xi^2)^2} + \frac{(z^2+1)(-2)(2z)}{(z+\xi)^2(z^2+\xi^2)^3}\right]_{z=\xi}$$

$$= \left(\frac{2\xi}{4\xi^2\, 4\xi^6} - \frac{2(\xi^2+1)}{8\xi^3\, 4\xi^4} - \frac{2.2(\xi^2+1)\xi}{4\xi^2 . 8\xi}\right)\Bigg|_{z=\xi}$$

use $\xi^8 = 1$

$$= \left(\frac{\xi^3}{8} - \frac{2(\xi^2+1)\xi}{32} - \frac{4(\xi^2+1)\xi}{32}\right)$$

$$= \left(\frac{\xi^3}{8} - \frac{3\xi^3+3\xi}{16}\right)$$

$$= \frac{-\xi^3-3\xi}{16}.$$

$\therefore$ Sum of residues in upper half plane

$$= \frac{-z_1^3-3z_1}{16} - \frac{z_2^3-3z_2}{16}$$

$z_1 = e^{i\pi/4} = \dfrac{1+i}{\sqrt{2}}$

$z_2 = e^{3i\pi/4} = \dfrac{-1+i}{\sqrt{2}}$

$$= \frac{-iz_1-3z_1}{16} + \frac{iz_2-3z_2}{16}$$

$z_1^3 = iz_1 \quad z_3^2 = -iz_2$

$$= \frac{i(z_2-z_1)-3(z_1+z_2)}{16} = \frac{-2i-6i}{16\sqrt{2}} = -\frac{i}{2\sqrt{2}}$$

$\therefore$ Given Integral $= \dfrac{1}{2} \times 2\pi i \times \left(\dfrac{-i}{2\sqrt{2}}\right) = \dfrac{\pi}{2\sqrt{2}}$

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