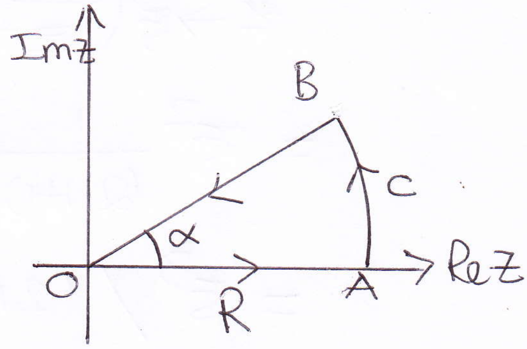
§§7.3
Q10.

To compute the value of integral

$$I = \int_0^\infty \frac{x^2}{x^{2N+1}+1}\, dx \quad , \quad N > 1.$$

using contour integration.

<u>Solution:</u> Use a sector contour in upper half plane. See example [2] of §§7.2. We take contour to be OACBO as shown in figure below and find suitable value of α to be used.



If R is the radius of the sector then

$$\int_{OA} \frac{z^2}{z^{2N+1}+1}\, dz = \int_0^R \frac{x^2\, dx}{x^{2N+1}+1} = I$$

$$\int_{AB} \frac{z^2}{z^{2N+1}+1}\, dz \to 0 \text{ as } R \to \infty \quad \text{(use Darboux Theorem)}$$

$$\int_{OB} \frac{z^2}{z^{2N+1}+1}\, dz = \int_0^R \frac{(x^2 e^{2i\alpha})(e^{i\alpha}\, dx)}{e^{(2N+1)i\alpha}\, x^{2N+1} + 1} \qquad z = xe^{i\alpha} \text{ along } OB$$

$$= e^{3i\alpha} \int_0^\infty \frac{x^2\, dx}{x^{2N+1}+1} \qquad \because \text{ We choose } \alpha = 2\pi/(2N+1)$$

$$\therefore \oint_{OACBA} z^2/(z^{2N+1}+1)\, dz = (1 - e^{3i\alpha})\, I \quad \left\{ \begin{array}{l} \text{in limit} \\ R \to \infty \end{array} \right.$$

The left hand side equals $2\pi i \times$ sum of residues at poles enclosed.

The denominator has zeros at points given by

$$z^{2N+1} = -1 \Rightarrow z = e^{i\pi/(2N+1) + 2m\pi i/(2N+1)}$$
$$m = 0, 2, \ldots, 2N$$

With our choice of α, only one pole at $\xi = e^{i\pi/(2N+1)}$ lies inside the contour for large R.

$$\text{Residue } \left.\frac{z^2}{z^{2N+1}+1}\right|_{z=\xi}$$

$$= \lim_{z\to\xi} \frac{(z-\xi)\, z^2}{(z^{2N+1}+1)}$$

$$= \lim_{z\to\xi} \left(\frac{z-\xi}{z^{2N+1}+1}\right) \times z^2$$

Use product of limits equals to the limit of products whenever latter exist.

$$= \xi^2 \frac{1}{(2N+1)\,\xi^{2N}}$$

$$= \xi^3 / (2N+1)\, \xi^{2N+1}$$

use $\xi^{2N+1} = -1$

$$= -e^{3i\alpha}/(2N+1).$$

note $\xi = e^{i\pi/(2N+1)}$
$= e^{i\pi\alpha/2}$

$$\therefore \left(1 - e^{3i\frac{\alpha}{2}}\right) I = 2\pi i \frac{(-1) e^{3i\alpha}}{(2N+1)}$$

$$I = \frac{2\pi i}{(2N+1)} \frac{(-1)}{(e^{-3i\alpha/2} - e^{i3\alpha/2})}$$

$$= \frac{2\pi i}{(2N+1)} \frac{1}{(-2i)} \frac{1}{\sin(3\alpha/2)}$$

$$I = \frac{\pi}{(2N+1)} \operatorname{cosec}\left(\frac{3\pi}{(2N+1)}\right)$$

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