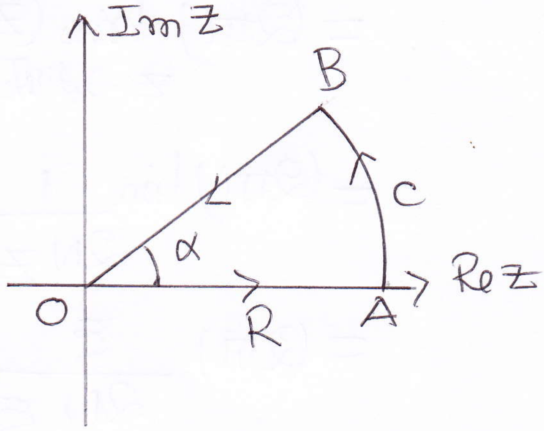
§§73 Q11

To compute the value of integral

$$I = \int_0^\infty \frac{x^4}{x^{2N}+1}\,dx$$

using contour integration.

Solution: As in Q[10], use a sector contour in upper half plane. We take the contour to be OACBO shown in figure below and find a suitable value of α to be used. Then we have



$$\lim_{R\to\infty}\int_{OA} \frac{z^4}{z^{2N}+1}\,dz = \int_0^\infty \frac{x^4}{x^{2N}+1}\,dx$$

$$\lim_{R\to\infty}\int_{BO} \frac{z^4}{z^{2N}+1}\,dz = -\int_{OB} \frac{z^4}{z^{2N}+1}\,dz$$

and

$$\lim_{R\to\infty}\int_A^B \frac{z^4}{z^{2N}+1}\,dz \longrightarrow 0 \text{ as } R\to\infty \quad \text{(Use Darboux Thm)}$$

$$\therefore \oint_{OACBO} \frac{z^4}{z^{2N}+1}\,dz = \int_{OA} - \int_{OB} \frac{z^4}{z^{2N}+1}\,dz$$

$$= -\int_0^R \frac{x^4 e^{i4\alpha}}{x^{2N}+1} \frac{e^{i\alpha}dx}{} dx + \int_0^R \frac{x^4 dx}{x^{2N}+1} dx$$

Choose $\alpha \times 2N = 2\pi$

$$= (1-e^{5i\alpha})\int_0^R \frac{x^4\,dx}{x^{2N}+1}$$

lim $R\to\infty$ is understood

$$= (1-e^{5i\alpha})\, I$$

$\therefore (1-e^{5i\alpha}) I = \oint_{OACBO} \frac{z^4}{z^{2N}+1} dz$

With our choice of α only one pole, at $z = e^{i\pi/2N} \equiv \xi$ $= e^{i\alpha/2}$ is enclosed inside the contour. Hence

$$(1-e^{5i\alpha}) I = 2\pi i \text{ Residue } \left.\frac{z^4}{z^{2N}+1}\right|_{z=e^{i\alpha/2}}$$

$$= (2\pi i) \lim_{z\to e^{i\pi/2}} \frac{(z-e^{i\alpha/2})}{z^{2N}+1} \times z^4$$ ← Simple pole at $z = \xi = e^{i\alpha/2}$

$$= (2\pi i) \lim \frac{1}{2N z^{2N}} \times z^4$$ ← Product of limits equals limit of products when separate limits exist

$$= (2\pi i) \frac{\xi}{2N \xi^{2N+1}} \xi^4$$ ← use $\xi^{2N+1} = -1$

$$= 2\pi i \frac{\xi^5}{(-2N)}$$

$$\therefore (1-e^{5i\alpha}) I = -\frac{2\pi i}{2N} e^{i5\alpha/2}$$

$$I = \frac{2\pi i}{2N} \frac{e^{5i\alpha/2}}{(1-e^{5i\alpha})}$$

$$= \frac{2\pi i}{2N} \frac{1}{(e^{-5i\alpha/2} - e^{5i\alpha/2})}$$

Recall $\alpha = \pi/N$

$$= \frac{2\pi i}{2N} \frac{1}{2i \sin \pi\alpha/2}$$

$$= \frac{\pi}{2N} \operatorname{cosec}(5\pi/2N)$$

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