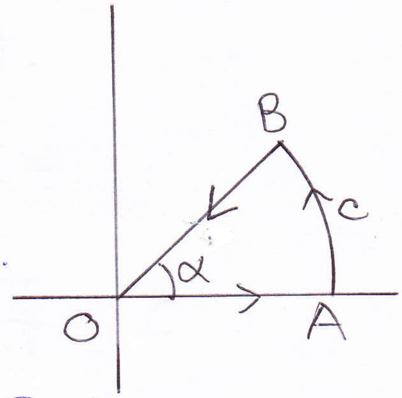
§§7.3
Q 12

To compute the value of integral

$$I = \int_0^\infty \frac{dx}{x^N+1}, \quad N>0.$$

using contour integration.

<u>Solution</u>: This integral is very similar to those in Q[10] and Q[11]. Use the contour shown in figure with



$$e^{iN\alpha} = 1 \Rightarrow \alpha = 2\pi/N.$$

The integral along the circular arc ACB vanishes as $R \to \infty$ (Darboux Thm).

$$\therefore \oint_{OACBO} \frac{dz}{z^N+1} = \oint_{OA} + \oint_{BO} \frac{dz}{z^N+1}$$

← lim $R\to\infty$ is implied

$$= \int_{OA} \frac{dz}{z^N+1} - \int_{OB} \frac{dz}{z^N+1}.$$

$$= \int_0^\infty \frac{dx}{x^N+1} - \int \frac{e^{i\alpha}\,dx}{x^N+1}$$

← because along OB $z = xe^{i\alpha}$ $x>0$
OA $z = x$

$$= (1-e^{i\alpha})\, I$$

There is only one pole enclosed by the contour. The location of the pole is given by

$$z^N+1 = 0 \Rightarrow z = e^{i\pi/N}, \ldots$$

Residue $\left.\frac{1}{z^N+1}\right|_{z=e^{i\pi/N}}$ $\lim\left(\frac{z-e^{i\pi/N}}{z^N+1}\right) = \left.\frac{1}{Nz^{N-1}}\right| = (-e^{i\pi/N})$

$$\therefore\ (1-e^{i\alpha})\, I = 2\pi i \times (-1) \times e^{i\pi/N}$$

or $$I = \frac{(-2\pi i)\, e^{i\pi/N}}{N(1-e^{2\pi i/N})}$$

Recall $\alpha = \frac{2\pi}{N}$

$$= \frac{(-2\pi i)}{(e^{-i\pi/N} - e^{i\pi/N})}$$

$$= \frac{\pi}{N} \cdot \frac{1}{\sin \pi/N}$$

$$I = \frac{\pi}{N} \operatorname{cosec}(\pi/N).$$

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