§§7.3
Q13

Compute the value of integral

$$\int_0^\infty \frac{x^{2M}}{x^{2N}+1}\,dx \qquad N>M>0$$

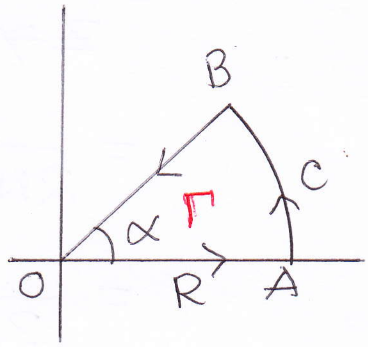
using contour integration.

<u>Solution</u>; Let the required integral be denoted by

$$I = \int_0^\infty \frac{x^{2M}}{x^{2N}+1}\,dx$$

To compute this integral we set up contour integral J

$$\oint_\Gamma \frac{z^{2M}}{z^{2N}+1}\,dz \equiv J_\Gamma$$



where Γ is sector OACBO, and the angle α is taken to be $\alpha = \frac{2\pi}{2N} = \pi/N$.

$$J_\Gamma = \int_{OA} + \int_{ACB} + \int_{BO} \frac{z^{2M}}{z^{2N}+1}\,dz$$

limit $R\to\infty$ will be taken. In this limit the integral along ACB vanishes (use Darboux theorem)

$$\therefore \lim J_\Gamma = \int_{OA} \frac{z^{2M}}{z^{2N}+1}\,dz - \int_{OB} \frac{z^{2M}}{z^{2N}+1}\,dz$$

$$= \int_0^\infty \frac{x^{2M}}{x^{2N}+1}\,dx - \int_0^\infty \frac{x^{2M} e^{(2M+1)i\alpha}}{x^{2N}+1}\,dx$$

$$= \left(1-e^{(2M+1)i\alpha}\right) I$$

The integral J_n can be computed by using residue theorem. One pole at $z = e^{i\pi/2N} (\equiv \xi)$ is enclosed inside the contour. The residue of the integrand at $z = \xi$ is

$$\mathrm{Res} \left.\frac{z^{2M}}{z^{2N}+1}\right|_{z=\xi} = \lim_{z\to\xi} \frac{z^{2M}}{z^{2N}+1}$$

$$= \lim_{z\to\xi} \frac{z-\xi}{z^{2N}+1} \cdot z^{2M}$$

$$= \left.\frac{1}{2N \cdot z^{2N-1}}\right|_{z=\xi} \xi^{2M}$$

$$= \frac{\xi^{2M+1}}{2N(-1)} = \frac{e^{i\alpha/2 \times (2M+1)}}{-2N}$$

$$\therefore J_n = (2\pi i) \frac{e^{i\alpha(2M+1)/2}}{(-2N)}$$

$$\therefore I = \frac{(2\pi i)\, e^{i\alpha(2M+1)/2}}{(-2N)} \times \frac{1}{(1 - e^{i\alpha(2M+1)})}$$

$$= \frac{2\pi i}{2N} \frac{e^{i\alpha(2M+1)/2N}}{(e^{i\alpha(2M+1)} - 1)}$$

$$= \frac{\pi i}{N} \cdot \frac{1}{2i \sin\alpha(2M+1)/2N}$$

$$\therefore I = \frac{\pi}{2N} \cdot \mathrm{cosec}\left[\frac{(2M+1)\pi}{2N}\right]$$

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