§§7.3
Q14

Compute the integral

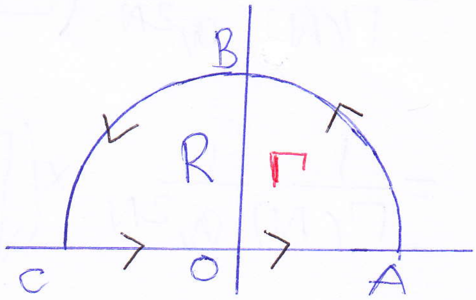
$$I = \int_0^{\infty} \frac{dx}{(p+qx^2)^N}, \quad p, q > 0.$$

using contour integration

The integrand is an even function of x. This integral can be computed by writing it as

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(p+qx^2)^N}$$

and adding a semicircular contour in the upper half plane. Therefore consider the contour Γ shown in figure.



$$\oint_\Gamma \frac{dz}{(p+qz^2)^N}$$

$$= \int_{-R}^{R} \frac{dx}{(p+qx^2)^N} + \int_{ABC} \frac{dz}{(p+qz^2)^N}$$

$\int_{ABC}$ goes to zero in limit $R \to \infty$

$$\lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{dx}{(p+qx^2)^N} = 2I$$

$$\therefore I = \frac{1}{2}\oint_\Gamma \frac{dz}{(p+qz^2)^N}$$

The integrand has only one pole inside Γ. The pole is at $z = i\sqrt{p/q}$. This is a pole of order N.

The residue is easy to calculate

$$\text{Res}\,\frac{1}{(p+qz^2)^N} = \frac{1}{(N-1)!}\,\frac{d^{N-1}}{dz^{N-1}}\,\frac{(z-i\sqrt{p/q})^N}{(p+iqz^2)^N}$$

We write $(p+qz^2)^N = q^N\,(z+i\sqrt{p/q})^N\,(z+i\sqrt{p/q})^N$

to get

$$\text{Res}\,\frac{1}{(p+qz^2)^N} = \frac{1}{(N-1)!}\,\frac{d^{N-1}}{dz^{N-1}}\,\frac{1}{q^{2N}}\,\frac{1}{i^{2N}}\left(\frac{1}{z+i\sqrt{p/q}}\right)^N\Bigg|_{z=i\sqrt{\frac{p}{q}}}$$

$$= \frac{1}{(N-1)!}\left(\frac{1}{q^{2N}}\right)\frac{1}{i^{2N}}\,\frac{d^{N-1}}{dz^{N-1}}\,\frac{1}{(z-i\sqrt{p/q})^N}$$

$$= \frac{1}{\Gamma(N)}\,\frac{1}{q^{2N}}\,\frac{1}{i^{2N}}\,\frac{N(N+1)\cdots(2N-1)(-1)^{N-1}}{(z-i\sqrt{p/q})^{2N-1}}\Bigg|_{z=\sqrt{\frac{p}{q}}\,i}$$

$$= \frac{1}{\Gamma(N)}\,\frac{1}{q^{2N}}\times\left(\frac{\sqrt{q}}{\sqrt{p}}\right)^{2N-1}\times\frac{1}{i^{2N}}\left(\frac{1}{2i}\right)^{2N-1}$$

$$\times\ \frac{(2N-1)\times(2N-3)\cdots}{(2N)\times(2N-2)\cdots}$$

← collect odd and even terms in the product $(2N-1)(2N-2)\cdots$

$$= \frac{1}{\Gamma(N)}\sqrt{\frac{p}{q}}\,\frac{1}{p^{2N}}\,\frac{1}{(2)^{2N-1}}\times\left(\frac{1}{i}\right)$$

$$\times\ \Gamma(N-1/2)\times 2^{2N-1}\times\frac{1}{\sqrt{\pi}}$$

$$\therefore\ I = \sqrt{\frac{\pi p}{q}}\,\frac{\Gamma(N-1/2)}{\Gamma(N)}\times\frac{1}{2p^N}$$

$$= \frac{1}{2}\sqrt{\frac{\pi p}{q}}\,\frac{1}{p^N}\,\frac{\Gamma(N-1/2)}{\Gamma(N)}$$

$\boxed{\text{April 23, 2017}}$