QM-18 Lecture Notes
Scattering Theory
Partial wave expansion

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§1 Introduction

We shall take up the method of partial waves for calculating cross section. This method, applicable to the spherically symmetric potentials, turns out to be particularly useful for short range potentials at low energies. We shall, therefore, restrict our attention to the spherically symmetric potentials $V(r)$ which for large r go to zero faster than $1/r^2$, *i.e.*, $r^2V(r) \to 0$.

§2 Partial Wave Expansion of Plane Waves

The free particle Schrodinger equation in three dimensions

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi \tag{1}$$

can be solved by separating the variables in the cartesian as well as the spherical polar coordinates. Writing the above equations as

$$\nabla^2\psi + k^2\psi = 0, \tag{2}$$

where

$$k^2 = \frac{2\mu E}{\hbar^2}, \tag{3}$$

solution of the free particle Schrodinger equation in the cartesian coordinates is given by

$$\psi_{\vec{k}}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r}). \tag{4}$$

There are infinite number of solutions, one for each value of the wave vector $\vec{k} = (k_1, k_2, k_3)$. If we write $k = |\vec{k}|$ and $\vec{k} = k\hat{n}$, the magnitude k is fixed for a fixed energy but the direction $\hat{n}$ can be arbitrary. Thus, for each energy value $E > 0$ the degeneracy is infinite, there being one solution for each $\hat{n}$, corresponding to the direction of propagation of the particle. Note that these solutions are also eigenvectors of the three momentum operators $\hat{\vec{p}}$ which together with the free particle Hamiltonian form a complete commuting set of operators.

The free particle Schrodinger equation Eq.(2) can also be solved by separating variables in spherical polar coordinates and we get the solution

$$\psi_{E\ell m}(\vec{r}) = j_\ell(kr)Y_{\ell m}(\theta,\phi) \tag{5}$$

For a given energy E, again one finds that there are infinite number of solutions, one for each value of ℓ and m, here $\ell = 0, 1, 2, \cdots$ and m can take integral values from $-\ell$ to ℓ. The eigenvectors in Eq.(5) are also simultaneous eigenvectors of $\vec{L}^2$ and $\vec{L}_z$ which together with the free particle Hamiltonian form another set of commuting operators.

Summarizing the above results, we see that for $E > 0$ there are two infinite sets of eigenfunctions given by

Simultaneous energy, momentum eigenfunctions

$$\left\{\psi_{\vec{k}}(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})\middle|\vec{k} = k\hat{n}, \quad \text{all} \quad \hat{n}\right\} \tag{6}$$

Simultaneous energy, angular momentum eigenfunctions

$$\left\{\psi_{E\ell m}(k\vec{r}) = j_\ell(kr)Y_{\ell m}(\theta,\phi)\middle|\ell = 0,1,2,\cdots; m = -\ell, -\ell+1, \cdots, \ell\right\} \tag{7}$$

The two sets of eigenfunctions in Eq.(6) and Eq.(7) form two different bases for solutions of given energy E; every function in Eq.(6) can be expanded in terms of the functions in the set Eq.(7) and also, every function in Eq.(7) can be expanded in terms of the functions in the set Eq.(6). Therefore, we can write the plane wave solutions, Eq.(6), as linear combinations of the spherical wave solutions Eq.(7).

$$\exp(i\vec{k}\cdot\vec{r}) = \sum_{\ell m} C_{\ell m} j_\ell(kr) Y_{\ell m}(\theta,\phi) \tag{8}$$

We shall now consider a special case when $\vec{k}$ is along the $z-$ axis $\vec{k} = (0,0,k)$ and $\vec{k}\cdot\vec{r} = kr\cos\theta$. The left hand side depends only on $\cos\theta$ and is independent of ϕ.Hence only

$m = 0$ terms will be present in the right hand side giving $C_{\ell m} = 0$ if $m \neq 0$. Also noting that

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) \tag{9}$$

the Eq.(8) takes the form

$$\exp(ikr\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell j_\ell(kr) P_\ell(\cos\theta) \tag{10}$$

We determine the coefficients A_ℓ by using orthogonality property of the Legendre polynomials.

$$\int_0^\pi P_\ell(\cos\theta) P_n(\cos\theta) \sin\theta d\theta = \frac{2}{2\ell+1} \delta_{\ell n}. \tag{11}$$

Therefore, we multiply Eq.(10) by $P_n(\cos\theta)\sin\theta$ and integrate over θ from 0 to π giving

$$\begin{aligned} &\int_0^\pi \exp(ikr\cos\theta) P_n(\cos\theta) \sin\theta d\theta \\ &= \sum_{\ell=0}^{\infty} A_\ell j_\ell(kr) \int_0^\pi P_\ell(\cos\theta) P_n(\cos\theta) \sin\theta d\theta \\ &= \sum_{\ell=0}^{\infty} A_\ell j_\ell(kr) \frac{2}{2\ell+1} \delta_{\ell n} \end{aligned} \tag{12}$$

or

$$\int_0^\pi \exp(ikr\cos\theta) P_n(\cos\theta) \sin\theta d\theta = \Big(\frac{2}{2n+1}\Big) A_n j_n(kr) \tag{13}$$

Changing the integration variable from θ to $t = \cos\theta$ in the left hand side we get

$$\int_{-1}^{1} \exp(ikrt) P_n(t) dt = \frac{2A_n j_n(kr)}{(2n+1)} \tag{14}$$

At this stage we should compute the integral on the left hand side and compare with the right hand side and get the value of A_n. Instead of trying to compute the integral exactly, we shall compute it for large r and compare the answer with the large r behaviour of the right hand side. The large r behaviour of the spherical Bessel function is known and is given by

$$j_n(kr) \longrightarrow \frac{1}{kr} \sin(kr - n\pi/2). \tag{15}$$

The large r asymptotic expansion of the the integral in the left hand side can be found by integrating by parts as follows.

$$\int_{-1}^{1} \exp(ikrt) P_n(t) dt$$

$$= \frac{1}{ikr} e^{ikrt} P_n(t)\Big|_{-1}^{1} - \frac{1}{ikr} \int_{-1}^{1} e^{ikrt} P_n'(t) dt \tag{16}$$

$$= \frac{1}{ikr}\Big(e^{ikr} - (-1)^n e^{-ikr}\Big) - \frac{1}{(ikr)^2} e^{ikrt} P_n'(t)\Big|_{-1}^{1} \tag{17}$$

$$+ O(\frac{1}{(ikr)^3}) \tag{18}$$

Thus as $r \to \infty$ we get

$$\begin{aligned}
\int_{-1}^{1} \exp(ikrt) P_n(t) dt \quad &\longrightarrow \quad \frac{1}{ikr}\left(e^{ikr} - (-1)^n e^{-ikr}\right) \\
&= \frac{e^{in\pi/2}}{ikr}\left(e^{ikr-n\pi/2} - e^{ikr-n\pi/2}\right) \\
&= \frac{2e^{in\pi/2}}{kr}\sin(kr - n\pi/2) \\
&= \frac{2e^{in\pi/2}}{kr}\cos(kr - (n+1)\pi/2) \qquad (19)
\end{aligned}$$

Using Eq.(15) and Eq.(19) in Eq.(14) we get

$$A_n = (2n+1)e^{in\pi/2} = (2n+1)i^n \qquad (20)$$

Thus the expansion of the plane waves, Eq.(10) takes the form

$$\boxed{e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta)}. \qquad (21)$$

A formula similar to Eq.(21) can be written down when the plane wave propagates in a direction other than the z- axis. Let α, β be the polar angles of the direction of propagation $\hat{n}$ so that

$$\hat{n} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\beta) \qquad (22)$$

If the angle between $\vec{k}$ and $\vec{r}$ is Φ, the relation Eq.(21) assumes the form

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\Phi} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\Phi) \qquad (23)$$

Using the well known addition theorem for spherical harmonics

$$\frac{2\ell+1}{4\pi}P_\ell(\cos\Phi) = \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\alpha,\beta)Y_{\ell m}(\theta,\phi) \qquad (24)$$

in Eq.(23) we get

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr)Y^*_{\ell m}(\alpha,\beta)Y_{\ell m}(\theta,\phi) \qquad (25)$$

§3 Asymptotics of Radial Wave Function

For computing the cross section, we need to solve the energy eigenvalue problem

$$H\psi = E\psi \qquad (26)$$

for continuous energies relevant for scattering situations, there are an infinite number of solutions we need to pick up a solution satisfying the boundary condition

$$\psi(\vec{r}) \longrightarrow \exp(ikz) + f(\theta, \phi)\, \frac{e^{ikr}}{r} \tag{27}$$

where $f(\theta, \phi)$ is the scattering amplitude and $|f(\theta, \phi)|^2$ gives the cross section. For a spherically symmetric potential, the Schrodinger equation can be solved by separation of variables in polar coordinates giving the energy eigenfunctions in the form

$$u_E(r, \theta, \phi) = R_{E\ell}(r) Y_{\ell m}(\theta, \phi) \tag{28}$$

and the most general solution for a given energy is a superpostion

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} R_{E\ell}(r) Y_{\ell m}(\theta, \phi). \tag{29}$$

For a spherically symmetric potential, the operators $\vec{L}^2$ and L_z commute with H and are constants of motion. Thus if incident particle has a definite value of $\vec{L}^2$ and L_z, it continues to have the same values at all times and the problem of finding the scattering amplitude is solved by finding the amplitude for different values of $\vec{L}^2$ and L_z separately. More over, it is sufficient to find the solutions for $L_z = 0$. This is because if we select the $z-$ axis parallel to the incident momentum $\vec{k}_i$, the initial value of L_z will be equal to zero and will remain zero at all times. Recall that

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

and eigenvectors with zero eigenvalue for L_z must be independent of ϕ. Hence is sufficient to restrict the sum over m in Eq.(29) to $m = 0$ terms only and the sum then reduces to the following form

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} C_\ell R_{E\ell}(r) P_\ell(\cos\theta). \tag{30}$$

where $Y_{\ell 0}(\theta, \phi) = \mathrm{const} P_\ell(\cos\theta)$ has been used. It is therefore sufficient to solve the radial equation

$$-\frac{\hbar^2}{2\mu} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR_{E\ell}(r)}{dr} \right) + \left(E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu} \right) R_{E\ell}(r) = 0 \tag{31}$$

and the knowledge of the asymptotic behaviour of the solution for the radial wave function $R_\ell(r)$ gives the cross section. The method of partial waves thus consists of the following steps.

$\boxed{1}$ Solve the radial equation for $R_\ell(r)$.

$\boxed{2}$ Find the asymptotic behaviour of the radial wave function $R_\ell(r)$.

$\boxed{3}$ Relate the large r behaviour of the radial wave function to the scattering amplitude and hence to the cross section.

We assume that the potential has a finite range and that the potential $V(r) \to 0$, as $r \to \infty$, faster than $\frac{1}{r^2}$ *i.e.* $r^2 V(r) \to 0$ as $r \to \infty$. Thus for large r, $V(r)$ can be neglected as compared

to $\ell(\ell+1)\hbar^2/(2\mu r^2)$ term in the radial equation

$$\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{E\ell}(r)}{dr}\right) + \left(E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)R_{E\ell}(r) = 0 \tag{32}$$

which for large r assumes the form

$$\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{E\ell}(r)}{dr}\right) + \left(E - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)R_{E\ell}(r) \approx 0 \tag{33}$$

or

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_{E\ell}(r)}{dr}\right) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)R_{E\ell}(r) \approx 0 \tag{34}$$

The most general solution of this equation is a linear combination of the spherical Bessel functions $j_\ell(kr)$ and $n_\ell(kr)$

$$R_{E\ell}(r) \overset{r\to\infty}{\approx} A_\ell j_\ell(kr) + B_\ell n_\ell(kr) \tag{35}$$

Note even though the radial equation looks like a free radial particle equation, the combination Eq.(35) is approximate solution for large r only, *where as for the free particle solution is $j_\ell(kr)$ for all r, a term $n_\ell(kr)$ is absent in the free particle solution.* Since we are interested only in large r behaviour of the radial wave function, we use the following asymptotic expansions for spherical the Bessel functions.

$$j_\ell(kr) \approx \frac{1}{kr}\cos(kr - \ell(\ell+1)\pi/2) \tag{36}$$
$$n_\ell(kr) \approx \frac{1}{kr}\sin(kr - \ell(\ell+1)\pi/2) \tag{37}$$

Using Eq.(36) and Eq.(37) we get the following results for the asymptotic form of the radial wave functions.

Free Particle Solution

The free particle radial wave function is

$$\begin{aligned} R_{E\ell}(r) &= C_\ell\, j_\ell(kr) \qquad (38)\\ &\approx \frac{C_\ell}{kr}\cos(kr - (\ell+1)\pi/2) \qquad (39)\end{aligned}$$

Scattering Solution for $V(r) \neq 0$
for the radial wave function is

$$
\begin{aligned}
R_{E\ell}(r) &= A_\ell j_\ell(kr) + B_\ell n_\ell(kr) && (40)\\
&\approx \frac{1}{kr}\Big(A_\ell \cos(kr - (\ell+1)\frac{\pi}{2}) \\
&\qquad\qquad + B_\ell \sin(kr - (\ell+1)\frac{\pi}{2})\Big) && (41)\\
&= \frac{A'_\ell}{kr}\cos(kr - (\ell+1)\pi/2 + \delta_\ell) && (42)
\end{aligned}
$$

where we have defined

$$A'_\ell = \sqrt{A_\ell^2 + B_\ell^2}, \qquad \tan\delta_\ell = -B_\ell/A_\ell \tag{43}$$

Comparing the scattering solution for the potential Eq.(43) with the free particle solution Eq.(39), we find that the Eq.(42) is shifted by a phase δ_ℓ given by Eq.(43). This quantity δ_ℓ is called the **phase shift** for the ℓ^{th} partial wave. The phase shift is a function of energy, and of course the angular momentum ℓ, and carry all the information about the scattering for angular momentum ℓ.

§4 Relating Cross Section to Phase Shifts

In order to derive an expression for the scattering amplitude in terms of the phase shifts, we substitute the partial wave expansion for the plane waves

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta), \tag{44}$$

and that of the wave function corresponding to the scattering solution

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} C_\ell R_{E\ell}(r)P_\ell(\cos\theta) \tag{45}$$

with $R_{E\ell}(r)$ having an asymptotic expansion of the form

$$
\begin{aligned}
R_{E\ell}(r) &\approx \frac{A'_\ell}{kr}\cos(kr - (\ell+1)\pi/2 + \delta_\ell) && (46)\\
&\approx \frac{A'_\ell}{kr}\sin(kr - \ell\pi/2 + \delta_\ell) && (47)
\end{aligned}
$$

in the boundary condition to be required of the wave function for large r

$$\psi(\vec{r}) \longrightarrow \exp(i\vec{k}\cdot\vec{z}) + f(\theta,\phi)\,\frac{e^{ikr}}{r} \tag{48}$$

and collect the coefficients of $\exp(\pm ikr)$ in the the sides of Eq.(48). This gives us the following expression for the left and the right hand sides of Eq.(48).

Left hand side of Eq.(48)
The wave function is a superposition of the radial solution for different partial waves and we have

$$\begin{aligned}
\text{L.H.S.} &= \psi(\vec{r}) &(49)\\
&= \sum_{\ell=0}^{\infty} C_\ell R_{E\ell}(r) P_\ell(\cos\theta) &(50)\\
&\approx \sum_{\ell=0}^{\infty} A'_\ell P_\ell(\cos\theta) \frac{1}{kr} \sin(kr - \ell\pi/2 + \delta_\ell) &(51)\\
&= \sum_{\ell=0}^{\infty} A'_\ell P_\ell(\cos\theta) \frac{1}{2ikr} \left\{ e^{i(kr-\ell\pi/2+\delta_\ell)} - e^{-i(kr-\ell\pi/2+\delta_\ell)} \right\} &(52)\\
&= \frac{e^{ikr}}{2ikr} \left\{ \sum_{\ell=0}^{\infty} A'_\ell P_\ell(\cos\theta) e^{i\delta_\ell} \right\} \\
&\qquad - \frac{e^{-ikr}}{2ikr} \left\{ \sum_{\ell=0}^{\infty} A'_\ell P_\ell(\cos\theta) e^{i\ell\pi/2} e^{-i\delta_\ell} \right\} &(53)
\end{aligned}$$

Right hand side of Eq.(48)
We make use of the plane wave expansion in the right hand side to get

$$\begin{aligned}
&\sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta) + f(\theta) \frac{e^{ikr}}{r} \\
&\approx \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell \frac{1}{kr} \sin(kr - \ell\pi/2 + \delta_\ell) P_\ell(\cos\theta) + f(\theta) \frac{e^{ikr}}{r} \\
&= \frac{1}{2ikr} \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell \left\{ e^{i(kr-\ell\pi/2+\delta_\ell)} - e^{-i(kr-\ell\pi/2+\delta_\ell)} \right\} &(54)\\
&\qquad \times P_\ell(\cos\theta) + f(\theta) \frac{e^{ikr}}{r} \\
&= \frac{e^{ikr}}{2ikr} \left\{ 2ikf(\theta) + \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell P_\ell(\cos\theta) e^{-i\ell\pi/2} \right\} \\
&\quad - \frac{e^{-ikr}}{2ikr} \left\{ \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell P_\ell(\cos\theta) e^{i\ell\pi/2} \right\} &(55)
\end{aligned}$$

Since the exponentials e^{ikr} and e^{-ikr} are linearly independent functions, their coefficients in Eq.(48) must be equal giving

$$2ikf(\theta) + \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell P_\ell(\cos\theta) e^{-i\ell\pi/2} = \sum_{\ell=0}^{\infty} A'_\ell P_\ell(\cos\theta) e^{-i\ell\pi/2} e^{i\delta_\ell} \quad (56)$$

and

$$\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}P_{\ell}(\cos\theta)e^{i\ell\pi/2} = \sum_{\ell=0}^{\infty}A'_{\ell}P_{\ell}(\cos\theta)e^{i\ell\pi/2}e^{-i\delta_{\ell}} \tag{57}$$

Because the Legendre polynomials $P_{\ell}(\cos\theta)$ are linearly independent, their coefficients in the two sides of Eq.(57) must be equal. This gives

$$A'_{\ell} = (2\ell+1)i^{\ell}e^{i\delta_{\ell}} \tag{58}$$

Substituting Eq.(58) in Eq.(56) and noting $i^{\ell} = e^{i\ell\pi/2}$, we get

$$\begin{aligned} 2ikf(\theta) &= -\sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(\cos\theta) + \sum_{\ell=0}^{\infty}(2\ell+1)e^{2i\delta_{\ell}}P_{\ell}(\cos\theta) \\ &= \sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(\cos\theta)\left(e^{2i\delta_{\ell}}-1\right) \end{aligned} \tag{59}$$

Therefore

$$f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)P_{\ell}(\cos\theta)\left(e^{2i\delta_{\ell}}-1\right) \tag{60}$$

gives the scattering amplitude. The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{61}$$

and the total cross section is

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega}d\Omega \tag{62}$$

$$= \int_0^{\pi} |f(\theta)|^2\, 2\pi\sin\theta\, d\theta \tag{63}$$

The θ integration can be completed using orthogonality of the Legendre polynomials and one gets

$$\sigma_{\text{total}} = \left(\frac{4\pi}{k^2}\right)\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_{\ell} \tag{64}$$

§5 Phase Shifts for Square Well

Problem 1: Compute the phase shifts for a square well potential.

$$V(r) = \begin{cases} -V_0 & \text{if } r < R_0, \\ 0 & \text{if } r > R_0. \end{cases} \tag{65}$$

☺*Solution:* The radial equation for a spherically symmetric potential $V(r)$ is

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_\ell(r)}{dr}\right)+\left(E-V(r)-\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)R_\ell(r)=0 \tag{66}$$

For $r<R_0$, the potential is $-V_0$ and the most general solution of the radial equation is

$$R_\ell^{(I)}(r)=\alpha j_\ell(qr)+\beta n_\ell(qr) \qquad r<R_0 \tag{67}$$

where $q^2=2\mu(E+V_0)/\hbar^2$ and for $r>R_0$ the potential is zero and the most general solution is

$$R_\ell^{(II)}(r)=\alpha' j_\ell(kr)+\beta' n_\ell(kr), \qquad r>R_0, \tag{68}$$

where $k^2=2\mu E/\hbar^2$. Next we must impose regularity conditions on the solutions. Since $n_\ell(\rho)$ blows up as $\rho\to 0$, we demand that β in Eq.(67) must be zero. Next the radial solution and its first derivative must be continuous at all points, in particular at $r=R_0$. These two conditions give the restrictions

$$R^{(I)}(r)|_{r=R_0} = R^{(II)}(r)|_{r=R_0} \tag{69}$$

$$\left.\frac{dR^{(I)}(r)}{dr}\right|_{r=R_0} = \left.\frac{dR^{(II)}(r)}{dr}\right|_{r=R_0} \tag{70}$$

Taking into account of the form of the solution we get

$$\alpha j_\ell(qR_0)=\alpha' j_\ell(kR_0)+\beta' n_\ell(kR_0) \tag{71}$$

$$\alpha q j'_\ell(qR_0)=\alpha' k j'_\ell(kR_0)+\beta' k n'_\ell(kR_0) \tag{72}$$

Dividing Eq.(72) by Eq.(71) we get

$$\frac{qj'_\ell(qR_0)}{j_\ell(qR_0)}=\frac{\alpha' kj'_\ell(kR_0)+\beta' kn'_\ell(kR_0)}{\alpha' j_\ell(kR_0)+\beta' n_\ell(kR_0)} \tag{73}$$

o

$$\frac{qj'_\ell(qR_0)}{j_\ell(qR_0)}=\frac{kj'_\ell(kR_0)+k(\beta'/\alpha')n'_\ell(kR_0)}{j_\ell(kR_0)+(\beta'/\alpha')n_\ell(kR_0)} \tag{74}$$

Noting that the phase shift is given by $\tan\delta_\ell=-(\beta'/\alpha')$, we solve for (β'/α') to get

$$\tan\delta_\ell=\left[\frac{\frac{k}{q}\frac{j'_\ell(kR_0)}{n_\ell(kR_0}\frac{j_\ell(qR_0)}{j'_\ell(qR_0)}-\frac{n_\ell(kR_0)}{n_\ell(kR_0)}}{1-\frac{k}{q}\frac{j_\ell(qR_0)}{j'_\ell(qR_0)}\frac{n'_\ell(kR_0)}{n_\ell(kR_0)}}\right] \tag{75}$$

For $\ell=0$, using the expressions for the spherical Bessel functions in terms of sine and cosine functions, for $s-$ wave one can easily obtain

$$\tan\left(\delta_0(k)+kR_0\right)=\frac{k}{q}\tan(qR_0) \tag{76}$$

wave

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