Set Theory Lecture Notes
Real Number System

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We will not construct the real numbers from rationals. We take real numbers as undefined objects which satisfy certain axioms. Starting from these axioms, all familiar properties can be proved.

The axioms for real number system come in three groups.

1. The Field axioms
2. The Order axioms
3. The Completeness axiom, or the least upper bound axiom.

We assume that the set $\mathbb{R}$ of real numbers is given to us and also given to us is a set $\mathbb{P} \in \mathbb{R}$ of positive reals. We also assume that two binary operations $+$ and $\cdot$ are defined. We assume that $\mathbb{P}, \mathbb{R}+$ and $\cdot$ satisfy the following relations.

THE FIELD AXIOMS For all $x, y, z \in \mathbb{R}$, we have

(A1) $x + y = y + x$;

(A2) $(x + y) + z = x + (y + z)$;

(A3) $\exists 0 \in \mathbb{R}$ s.t. $x + 0 = x, \forall x \in \mathbb{R}$;

(A4) For each $x \in \mathbb{R}$, there exists a $v \in \mathbb{R}$ s.t. $x + v = 0$.
Such a v is called 'additive inverse' of x and is denoted by $-x$;

(A5) $x \cdot y = y \cdot x, \qquad \forall x, y \in \mathbb{R}$;

(A6) $(x \cdot y)z = x(y \cdot z), \qquad \forall x, y, z \in \mathbb{R}$

(A7) $\exists 1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $x.1 = x, \quad \forall x \in \mathbb{R}$;

(A8) $\forall x \in \mathbb{R}, \quad x \neq 0, \quad$ there exists $w \in \mathbb{R}$ s.t. $x\,w = 1$;
w will be called multiplicative inverse of x;

(A9) Distributive law: $x(y+z) = xy + zx$.

B. AXIOM OF ORDER The subset $\mathbb{P}$ of positive real numbers satisfies the following axioms.

(B1) $x, y \in \mathbb{P} \Rightarrow x + y \in \mathbb{P}$;

(B2) $x, y \in \mathbb{P} \Rightarrow x.y \in \mathbb{P}$;

(B3) $x \in \mathbb{P} \Rightarrow -x \notin \mathbb{P}$;

(B4) $x \in \mathbb{R} \Rightarrow x = 0$, or $x = \notin \mathbb{P}$, or $x \in \mathbb{P}$
i.e. $\mathbb{R} = -\mathbb{P} \cup \{0\} \cup \mathbb{P}$, where $-\mathbb{P}$ is the set $\{x : -x \in \mathbb{P}\}$.

Using the above axiom, the familiar properties of order relation $<$ can be proved, if we define

$$x < y \text{ to mean } y - x \in \mathbb{P}.$$

Thus we have
(B1) $\Rightarrow$ $x < y$ and $z < w \Rightarrow x + z < y + w$;
(B2) $\Rightarrow$ $0 < x < y$ and $0 < z < w \Rightarrow xz < yw$
(B4) $\Rightarrow$ If $x \in \mathbb{R}, y \in \mathbb{R}$, only one of the following holds .
$x < y$, or $x = y$, or $y < x$.
The last axiom given below the most important one.

C. COMPLETENESS AXIOM or THE LEAST UPPER BOUND AXIOM
Every nonempty set of real numbers which has an upper bound has a least upper bound.

References

[1] Royden, *Real Analysis*

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