Set Theory Lecture Notes
Countable and Uncountable Sets*

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Definition 1 *If there exists a 1-1 mapping of a set A onto a set B, we say that A can be put in a one to one correspondence with the set B and write $A \approx B$. In this case we also say that A is numerically equivalent to B, or equipollent to B.*

The relation of being numerically equivalent is seen to have the following properties

- *Reflexive property*: $A \approx A$ holds;
- *Symmetry property*: If $A \approx B$, then $B \approx A$;
- *Transitive property*: If $A \approx B$ and $B \approx C$, then $A \approx C$.

Therefore the relation $A \approx B$ is an equivalence relation,

Definitions Let J_n be the set of integers $\{1, 2, \ldots, n\}$.

(a) A set A is *finite*, if $\exists\, n$ such that $A \approx J_n$.

(b) A set A is *infinite*, if there does not exist integer n s.t. $A \approx J_n$, *i.e.* if A is not a finite set.

(c) A set A is *countable*, if it can be put in a one to one correspondence with the set $\mathbb{Z}^+$ of positive integers (*i.e.* $A \approx \mathbb{Z}^+$).

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(*d*) A set A is *at most countable* if it can be put in a one to one correspondence with $\mathbb{Z}^+$, or if it is finite. In other words a set is *most countable*, if it is finite or countable.

(*e*) A set is *uncountable*, if it is infinite and is not countable.

)|(**Short Examples 1** *We have following examples.*

(1a) The set of all positive integers $\mathbb{Z}^+$ *is countable because the map*

$$f : \mathbb{Z}^+ \to \mathbb{Z}^+$$

can be chosen to be the identity map and it will be one to one and onto.

(1b) The set of all integers is countable. We display a map which one-one and onto.

$$\begin{array}{lllllllll} \mathbb{Z} & \longrightarrow & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \ldots \\ & & \downarrow & & & & & & & \\ \mathbb{Z}^+ & \longrightarrow & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots. \end{array}$$

This map can also be given as a function $f : \mathbb{Z} \to \mathbb{Z}^+$

$$f(n) = \begin{cases} 2n+1 & if\, nispositive \\ -2n & if\, nisnegative \end{cases} \qquad (1)$$

(1c) The set P *of all perfect square integers is countable.*

$$P = \{1, 4, 9, 16, 25, 36, \ldots\}$$

and the map $f : P \longrightarrow \mathbb{Z}^+$ *given by* $f(n^2) = n$ *is one-one onto.*

(1d) The interval (0,1) is uncountable. So are all intervals $(a, b), (a, b], [a, b), [a, b], a \neq b$ *uncountable.*

Remark: *The elements of a countable set can be arranged in form of a sequence. It makes sense to write*

$$A = \{\alpha - 1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots\}$$

when the set A is countable but not when A is uncountable.

Theorem 1 *Every infinite subset of a countable set is countable.*

Remarks:

(i) If A *is countable and* $B \approx A$*, then* B *is countable.*

(ii) If $C \subseteq A$ *and* C *is infinite and* A *is countable. then the theorem* (1) *implies that* C *is countable. Thus if* X *is an infinite set and a map* $f : X \longrightarrow A$ *is one to one, but*

(ii) The theorem (1) *shows that the countable sets are in some sense represent smallest infinity.*

Theorem 2 $\mathbb{Z}^+ \times \mathbb{Z}^+$ *is countable*

Proof: $\forall$ *pairs* (m, n) *associate an integer* $2^m \times 3^m$*. This then is a one -one mapping of* $\mathbb{Z}^+ \times \mathbb{Z}^+$ *into the the set* $\mathbb{Z}^+$*. Hence the set* $\mathbb{Z}^+ \times \mathbb{Z}^+$ *is countable.*

Theorem 3 *If* P *and* Q *are countable sets, then the set* $P \times Q$ *is countable.*

The sets P and Q can e written as

$$P = \{p_1, p_2, \ldots\}, \qquad Q = \{q, q_2, \ldots\} \tag{2}$$

Hence

$$P \times Q = \{(p_m, q_n) \big| p_m \in P, q_n \in Q\}$$

Define a mapping $f : P \times Q \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$ by

$$f(p_m, q_n) = (m, n).$$

Therefore $P \times Q \approx \mathbb{Z}^+ \times \mathbb{Z}^+$.and by theorem (3) $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. Hence $P \times Q$ is countable.

Example *The set of all rationals is countable.*

Theorem 4 *Direct product of a finite number of countable sets is countable. Thus if each of the sets $A_1, A_2, \ldots, A_n$ is countable, then the set*

$$A_1 \times A_2 \times \ldots \times A_n$$

is countable.

Theorem 5 *If F is a countable collection of countable sets, then union of all sets in the collection is countable. Thus, if*

$$F = \{A_1, A_2, \ldots, A_n, \ldots\}$$

and if each A_k is countable set, then $S = \bigcup_{k=1}^{\infty} A_k$ is a countable set.

Remark: *The result of Theorem 5 remains true if countable is replaced by at most countable in the statement.*

)‖(***Short Examples 2*** *Some known countable sets are given below.*

- *(2a) Set of all positive integers Z^+ is countable;*
- *(2b) Set of all integers $\mathbb{Z}$ is countable;*
- *(2c) Set of all rationals $\mathbb{Q}$ is countable;*
- *(2d) The sets $\mathbb{Z} \times \mathbb{Z}, \mathbb{Z}^+ \times \mathbb{Z}^+, \mathbb{Q} \times \mathbb{Q}$ are all countable sets;*
- *(2e) Any infinite subset of one of the above sets is countable.*

How to prove an infinite set to be countable?

- *Find a 1-1 correspondence from the given set onto (or into) $\mathbb{Z}^+$ or any known countable set.*
- *By showing that it can be written as a direct product of a finite number of countable sets.*
- *By showing that the set is equal to) or a subset of a union of countable collection of countable (or at most countable) sets.*

Warning: *If S is a given set and if we construct a mapping $f : S \longrightarrow A$ to prove countability of S, we must show that (i) f is one to one and (ii) A must be countable.*

)|(***Short Examples 3*** *We list some examples of countable and uncountable sets.*

1. *Set of all circles in a plane with centres at rational coordinates, and rational radii is countable.*
2. *Set of all intervals with rational end points is countable*
3. *Set of all rationals is countable*
4. *Set of squares and rectangles with rational coordinates of the corners is countable.*
5. *If Q is a subset of P and if Q uncountable, the P is uncountable.*

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