QM-Lecture Notes 6.3
A Dicsussion of The Third Postulate of Quantum Mechanics*

Computation of Probability and Average Value

A. K. Kapoor
kapoor.proofs@gmail.com
akkhcu@gmail.com

§1 The Third Postulate

The third postulate has several parts. In this section we use the notation A to denote a dynamical variable, $\hat{A}$ will denote the corresponding hermitian operator. The eigenvalues will be real and it is assumed that the eigenvectors have been *chosen* to be orthonormal. If we denote the eigenvalues of $\hat{A}$ by $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ and the corresponding orthonormal eigenvectors by $|u_1\rangle, |u_2\rangle, \ldots, |u_k\rangle, \ldots$ then we have

$$A|u_k\rangle = \alpha_k|u_k\rangle, \qquad \langle u_k|u_m\rangle = \delta_{km}. \tag{1}$$

1. The first part of the third postulate says that the only outcome of a measurement of a dynamical variable A is one of the eigenvalues of the corresponding operator $\hat{A}$. Thus, if an experiment to measure a dynamical variable A is performed, the result must be only one of the eigenvalues α_k. In particular, an answer different, from every eigenvalue, cannot be the outcome of measurement of A.

2. The next part of the postulate tells that if a system is represented by one of the eigenvectors $|u_n\rangle$, a measurement of the dynamical variable A will give the corresponding eigenvalue α_n.

3. The third postulate also tells us about the outcome of measurement of A when the state vector $|\psi\rangle$ is not an eigenvector of $\hat{A}$. In this case, the result is some times one eigenvalue and sometimes another. We cannot predict the result of a single measurement fully. When repeated measurements are made, different eigenvalues α_n will be obtained with different probabilities p_n which can be predicted.

   To compute the probabilities p_n, we first expand the state vector $|\psi\rangle$ in terms of the eigenvectors of the operator $\hat{A}$ corresponding to the dynamical variable A which is being measured and write

$$|\psi\rangle = \sum_k c_k|u_k\rangle. \tag{2}$$

* KApoor//QM-Lecs-06.tex; Ver 16.x   November 13, 2016

Then the probability p_k of getting value α_k is given by $|c_k|^2$. We continue to assume that the state vector $|\psi\rangle$ and the eigenvectors $|u_k\rangle$ are orthogonal ,*i.e*,,

$$\langle\psi|\psi\rangle = 1, \qquad \langle u_k|u_k\rangle = 1 \tag{3}$$

4. How do we compute the coefficients c_k in Eq.(11)? The eigenvectors of a hermitian operator are orthogonal and this helps in computing the coefficients. Taking scalar product of Eq.(10) with $|u_n\rangle$ gives

$$\langle u_k|\psi\rangle \quad = \quad \sum_n c_k\langle u_k|u_n\rangle = \sum_n c_n\delta_{kn} = c_k \tag{4}$$

   Note that, in the right hand side of Eq.(12), only the term with $k = n$ survives, all other terms where $n \neq k$ will vanish due to orthogonality property of the eigenvectors.

$$\therefore \qquad c_k = \langle u_k|\psi\rangle \tag{5}$$

   and

$$\boxed{p_k = |c_k|^2 = |\langle u_k|\psi\rangle|^2} \tag{6}$$

5. The Parseval relation

$$\langle\psi|\psi\rangle = \sum_k |c_k|^2 \tag{7}$$

   implies that

$$\sum_k |c_k|^2 = 1 \Longrightarrow \sum_k p_k = 1. \tag{8}$$

   if the state vector $|\psi\rangle$ is normalised, $\langle\psi|\psi\rangle = 1$. This suggests that the interpretation of expressions $|c_k|^2$ as probabilities p_k is consistent with the requirement that the sum of all probabilities be equal to unity. We shall call the coefficient c_k as the probability amplitude for obtaining a value α_k for the dynamical variable A when the system is in state $|\psi\rangle$.

As a consequence of the dual nature of matter and radiation, we have indeterminacy in the theoretical predictions. The origin of this indeterminacy can be traced to the superposition principle which in turn is needed to incorporate the wave nature of matter. In classical mechanics the result of measurement of position, momenta, and every other dynamical variables, can be fully predicted. This is no longer true in quantum theory. Here is a summary.

Remarks

- A *single* measurement of A does not lead to a definite answer when the *state vector* is not an eigenvector $\hat{A}$.

- In general, a result of a measurement of A must be one of the eigenvalues. The outcome of a *single* experiment is *indeterminate*, and the quantum theory is probabilistic by its nature in contrast to the classical theory which is deterministic. When measurement is repeated several times, we will sometimes get an eigenvalue α_j sometimes some other eigenvalue α_k, and only the probabilities of each outcome can be predicted.

- A simple consequence of the above discussion is that a measurement of a dynamical variable A will give a value α_m with probability 1 if and only if the state is represented by corresponding eigenvector $|u_m\rangle$.
- It must be remembered that some obvious changes, described later at the end of the next section, will be needed when the eigenvalues of $\hat{A}$ are continuous.
- Finally, we leave it as an exercise for you to convince yourself that the assumption about the probabilities, as stated above, is correctly contained in the the following statement. *Given that the system is in a state described by the state vector $|\psi\rangle$, the probability that it will be found in the state given by the vector $|\phi\rangle$ is equal to $|\langle\phi|\psi\rangle|^2$.*

§2 Probabilities and Average Values

Let A denote a dynamical variable and $\hat{A}$ the corresponding hermitian operator representing A in quantum mechanics.Let $|u_n\rangle$ be normalised eigenvector of $\hat{A}$ with eigenvalue α_n.

$$\hat{A}|u_n\rangle = \alpha_n\,|u_n\rangle \tag{9}$$

These eigenvectors, being eigenvectors of a hermitian operator, will satisfy the orthogonality relation

$$\langle u_m|u_n\rangle = \delta mn. \tag{10}$$

If several repeated measurements are made on a system with *state vector* $|\psi\rangle$, one would get α_k with probability

$$p_k = |c_k|^2, \text{ where } c_k = \langle u_k|\psi\rangle. \tag{11}$$

The average of results of measurements of A in the state $|\psi\rangle$, to be denoted by $\langle A\rangle_\psi$, will then be given by

$$\langle A\rangle_\psi = \sum_k p_k\alpha_k = \sum_k \alpha_k\,|c_k|^2. \tag{12}$$

We will now show that the above expression coincides with $\langle\psi|\hat{A}|\psi\rangle$. Without loss of generality, we may assume that the state vector $|\psi\rangle$ and the eigenvectors $|u_k\rangle$ are normalised. To prove this result we recall that c_k are the expansion coefficients

$$|\psi\rangle = \sum_k c_k|u_k\rangle \tag{13}$$

and make use of orthonormal property (10) to compute $\langle\psi|\hat{A}|\psi\rangle$.

$$\begin{aligned}\langle\psi|\hat{A}|\psi\rangle &= (\psi, \hat{A}\psi) &(14)\\ &= (\psi, \hat{A}\sum_k c_k u_k) = \sum_k c_k(\psi,\ \hat{A}\, u_k) &(15)\\ &= \sum_k c_k\,(\psi,\ \alpha_k\, u_k) = \sum_k c_k\,\alpha_k(\psi,\ u_k) &(16)\\ &= \sum_k c_k\,\alpha_k \bar{c}_k = \sum_k \alpha_k\,|c_k|^2. &(17)\end{aligned}$$

which is seen to be equal to the average $\langle A\rangle_\psi$ from Eq.(12). When the state vector $|\psi\rangle$ is not normalised, the average value will be given by

$$\langle A\rangle_\psi = \frac{\langle\psi|\hat{A}|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{18}$$

Case of continuous eigenvalues: So far our discussion has been restricted to the case when the eigenvalues of $\hat{A}$ are discrete. Now consider the case when the eigenvalues α of $\hat{A}$ are continuous and the corresponding eigenvectors $|\alpha\rangle$ are normalised to Dirac delta function

$$\langle\alpha|\alpha'\rangle = \delta(\alpha - \alpha'). \tag{19}$$

In this case the probability that a measurement of A will give a value in a small range α and $\alpha + d\alpha$ is equal to $|\langle\alpha|\psi\rangle|^2 d\alpha$. For probability of finding the result in between a and b, we would have the answer $\int_a^b |\langle\alpha|\psi\rangle|^2 d\alpha$.