GT-06 Solved Example
IRR in Product of Two IRRs

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Problem 1: The group of permutations on three objects has three classes The character table of the group is given to be

ν		$\Gamma^{(1)}$	$\Gamma^{(2)}$	$\Gamma^{(3)}$
1	C_1	1	1	2
3	C_2	1	-1	0
2	C_3	1	1	-1

Find the irreducible representations contained in

[1] $\Gamma^{(2)} \otimes \Gamma^{(2)}$

[2] $\Gamma^{(3)} \otimes \Gamma^{(3)}$

[3] $\Gamma^{(2)} \otimes \Gamma^{(3)}$

☺*Solution*: First we have to get the characters of the product representations. We denote the characters of the three product representations as $\chi^{(2)\times(2)}, \chi^{(3)\times(3)}, \chi^{(2)\times(3)}$.

The characters of the product of two representations are obtained by multiplying the corresponding components of the character vectors of the individual representations. Thus we get

$$\chi^{(2)\times(2)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad \chi^{(3)\times(3)} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}; \quad \chi^{(2)\times(3)} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}. \tag{1}$$

$$\tag{2}$$

The representation $\Gamma^{(2)} \otimes \Gamma^{(2)}$ coincides with $\Gamma^{(1)}$ and is irreducible.
The representation $\Gamma^{(2)} \otimes \Gamma^{(3)}$ coincides with $\Gamma^{(3)}$ and is irreducible.
Let us now look at the irreducible representation content of $\Gamma^{(3)} \otimes \Gamma^{(3)}$. Let us denote the character vector of this representation as χ.

$$\chi = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \tag{3}$$

To find the content, we take the dot product of character vector χ with the character vectors $\chi^{(\alpha)}, \alpha = 1, 2, 3$ of different irreducible representations. The dot product is computed by using

$$(\chi^{(\alpha)}|\chi) = \frac{1}{\nu(G)} \sum_k \nu_k \chi_k^{(\alpha)} \chi_k. \tag{4}$$

Here $\nu(G)$ is the order (number of elements) of the group. For the group S_3 the order is $\nu(G) = 6$. The number elements in a class C_k has denoted by ν_k. These numbers given in the first column of the character table. Thus we get the following results.
No. of times $\Gamma^{(1)}$ is contained in $\Gamma^{3\times3}$ is given by

$$(\chi^{(1)}|\chi) = \frac{1}{6}\big(1.1.4. + 3.1.0 + 2.1.1\big) = 1 \tag{5}$$

No. of times $\Gamma^{(2)}$ is contained in $\Gamma^{3\times3}$

$$(\chi^{(2)}|\chi) = \frac{1}{6}\big(1.1.4 + 3.(-1).0 + 2.1.1\big) = 1 \tag{6}$$

No. of times $\Gamma^{(1)}$ is contained in $\Gamma^{3\times3}$ is given by

$$(\chi^{(3)}|\chi) = \frac{1}{6}\big(1.2.4 + 3.0.0 + 2.(-1).1\big) = 1. \tag{7}$$

Therefore, we get the following decomposition

$$\Gamma^{(3)\times(3)} = \Gamma^{(1)} \dot{+} \Gamma^{(2)} \dot{+} \Gamma^{(3)}. \tag{8}$$