21.8.2016 Solution gt-que-02003

The number of irreducible representations of a group (finite group) is equal to the number of classes.

Therefore there are five irreducible representations of D_4 group.

Let the dimensions of the irreducible representations be $n_1, n_2 \cdots n_5$. Then

$$\sum_{\alpha=1}^{5} n_\alpha^2 = 8 \quad \text{(order of the group)}$$

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 8$$

This can be satisfied by $n_1 = n_2 = n_3 = n_4 = 1$ and $n_5 = 2$.

Thus four irreducible representations are one dimensional and one irreducible representation is two dimensional.

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Let ν_j $(j = 1, 2 \cdots 5)$, denote number of group elements of class C_j and $\chi_j^{(\alpha)}$ be the characters of the class C_j. Introduce

$$\lambda_j^{(\alpha)} = \chi_j^{(\alpha)} \nu_\alpha / n_\alpha$$

then $\lambda_j^{(\alpha)}$ follow the same multiplication rule as class multiplication rules.

$\therefore \chi_2 = \pm 2$.

Using IV we get

$\frac{1}{8}(4+4+\chi_3^2+\chi_4^2+\chi_5^2) = 1$

This gives $\chi_3 = \chi_4 = \chi_5 = 0$

Thus character table becomes.

ν_j		Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
1	$C_1=\{e\}$	1	1	1	1	2
1	C_2	1				-2^* ($^*+2$ discarded)
2	C_3	1				0
2	C_4	1				0
2	C_5	1				0

Charater Table Structure.

Every finite group has a one dimensional representation which assigns number 1 to each element and hence its trace $= \chi_j^{(1)} = 1$ for all classes. This gives the first column of the character table.

The identity element is represented by the identity matrix and corresponding trace $= \chi_1^{(\alpha)}$ for an IRR is the dimension of the IRR. This gives the first row of the table.

The class multiplication rules are given tobe

$$C_2^2 = C_1 \qquad C_1 C_j = C_j \ , \ j = 2, \dots 5$$

$$C_3^2 = 2C_1 + 2C_2 ; C_4^2 = 2C_1 + 2C_2 ; C_5^2 = 2C_1 + 2C_2$$

$$C_2 C_3 = C_3 ; \quad C_2 C_4 = C_5 ; \quad C_2 C_5 = C_5 \qquad (\mathrm{I})$$

$$C_3 C_4 = 2C_5 , \quad C_3 C_5 = 2C_4 \quad C_4 C_5 = 2C_3$$

$\nu_1 = 1, \nu_2 = 1., \nu_3 = 2, \nu_4 = 2, \nu_5 = 2$
are the number of group elements in
Classes $C_1, \dots C_5$.

Consider 2 dimensional IRR of D_4. $n_\alpha = 2$
Drop index α.

$$\lambda_j = \chi_j \nu_j / 2 \qquad (\mathrm{II})$$

$$\lambda_1 = \chi_1/2 \ , \ \lambda_2 = \chi_2/2 \ , \ \lambda_3 = \chi_3 , \lambda_4 = \chi_4 , \lambda_5 = \chi_5$$

The character vectors are normalized as

$$\frac{1}{8}\left(|\chi_1|^2 + |\chi_2|^2 + 2|\chi_3|^2 + 2|\chi_4|^2 + 2|\chi_5|^2\right) = 1 \qquad (\mathrm{IV})$$

The class C_1 ~~is~~ consists of $\{e\}$, the identity element only. Hence its character is equal to the dimension of the IRR

$$\therefore \chi_1 = 2 \qquad (\mathrm{V})$$

Next $C_2^2 = C_1 \Rightarrow \lambda_2^2 = \lambda_1$ or using (II)

$$\chi_2^2/4 = \chi_1/2 \Rightarrow \chi_2^2 = 4$$

In the character table

$\alpha \rightarrow$ column index corresponding to IRR Γ_α

$j \rightarrow$ row index corresponding to different classes C_j.

ν_j = number of elements in the class.

The two orthogonality theorems given

$$\sum_\alpha \sqrt{\frac{\nu_j}{\nu(G)}}\, \chi_j^{*(\alpha)}\, \chi_k^{(\alpha)} \sqrt{\frac{\nu_k}{\nu(G)}} = \delta_{jk}.$$

and $\sum_j \chi_j^{(\alpha)*} \chi_j^{(\beta)} = 0$ if $\alpha \neq \beta$.

$\therefore$ For reasons of orthogonality of columns (1) and col 5 (for Γ_1 and Γ_5) we must select $\chi_2 = -2$. and discard $\chi_2 = +2$.

Characters of one dimensional representations.
For a one dimensional representation Γ_α, $n_\alpha = 1$ and hence $\lambda_j = \chi_j \nu_j$

This gives

$\lambda_1 = \chi_1 = 1$, $\lambda_2 = \chi_2$, $\lambda_3 = 2\chi_3$, $\lambda_4 = 2\chi_4$
$\lambda_5 = 2\chi_5$

The numbers λ_j follow the same multiplication rule as classes C_j. The class multiplication rules on p2 give $C_2^2 = C_1 \Rightarrow \lambda_2^2 = 1 \Rightarrow \chi_2 = \pm 1$

also $4\chi_3^2 = 2 + 2\chi_2$

$= 4 \quad (\because C_3^2 = 2C_1 + 2C_2)$

$\chi_2 = -1$ can be seen to inconsistent with orthogonality

$\therefore \chi_3^2 = 1$

$C_2 C_3 = C_3 \Rightarrow \quad \chi_2(2\chi_3) = 2\chi_3$

$\Rightarrow \chi_2 = 1$

$\cancel{C_2 C_4 = C_4 \Rightarrow}$

$C_4^2 = 2C_1 + 2C_2 \Rightarrow \chi_4^2 = 1$

$C_5^2 = 2C_1 + 2C_2 \Rightarrow \chi_5^2 = 1$

Orthogonality property of col. α with the first col. gives

$$\nu_1 \chi_1 + \nu_2 \chi_2 + \nu_3 \chi_3 + \nu_4 \chi_4 + \nu_5 \chi_5 = 0$$

$$\chi_1 + \chi_2 + 2(\chi_3 + \chi_4 + \chi_5) = 0$$

or $\because \chi_1 = \chi_2 = 2$ we get

$$\chi_3 + \chi_4 + \chi_5 = 0.$$

This along $\chi_3^2 = \chi_4^2 = \chi_5^2 = 1$ gives:

Two of these are -1 and the third one is $+1$.

Therefore for one dimensional representations we have three solutions. $x_1 = x_2 = 1$

$x_3 = +1 \quad x_4 = -1 \quad x_5 = -1$

$x_3 = -1 \quad x_4 = +1 \quad x_5 = -1$

$x_3 = -1 \quad x_4 = -1 \quad x_5 = +1$

and the complete character table is

		$\Gamma^{(1)}$	$\Gamma^{(2)}$	$\Gamma^{(3)}$	$\Gamma^{(4)}$	$\Gamma^{(5)}$
$\nu_1 = 1$	C_1	1	1	1	1	2
$\nu_2 = 1$	C_2	1	1	1	1	-2
$\nu_3 = 2$	C_3	1	1	-1	-1	0
$\nu_4 = 2$	C_4	1	-1	1	-1	0
$\nu_5 = 2$	C_5	1	-1	1	1	0.