

GT-02 Solved Problem

Getting Character Table for D_4

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Abstract

This solution illustrates construction of the character table of a finite group. This requires an understanding of class multiplication and orthogonality of characters.

Problem 1: What is the number of irreducible representations of the D-4 group. Find their dimensions and construct the character table for the group. You may use the notation and information given below.

⊗ Notation for group elements

- Anticlockwise rotation about z - axis by angle $\pi/2 \rightarrow 4_z$.
- Anticlockwise rotation about z - axis by angle $\pi \rightarrow 4_z^2$.
- Anticlockwise rotation about z - axis by angle $3\pi/2 \rightarrow 4_z^3$.
- Reflection in a plane perpendicular to X - axis $\rightarrow m_x$
- Reflection in a plane perpendicular to Y - axis $\rightarrow m_y$
- Reflection in plane containing Z - axis and diagonal 13 $\rightarrow m_{13}$
- Reflection in plane containing Z - axis and diagonal 24 $\rightarrow m_{24}$

The classes are given to be

$$\begin{aligned} C_1 &= \{e\}. \\ C_2 &= \{4_z^2\} & C_3 &= \{4_z, 4_z^3\} \\ C_4 &= \{m_x, m_y\} & C_5 &= \{m_{13}, m_{24}\} \end{aligned}$$

Class multiplication rules:

$$\begin{aligned} C_2^2 &= C_1 & C_2 C_3 &= C_3 & C_2 C_4 &= C_4 & C_2 C_5 &= C_5 \\ C_3^2 &= 2C_1 + C_2 & C_3 C_4 &= 2C_5 & C_3 C_5 &= 2C_4 \\ C_4^2 &= 2C_1 + 2C_2 & C_4 C_5 &= 2C_3 \\ C_5^2 &= 2C_1 + 2C_2 \end{aligned}$$

Fig. 1

The number of irreducible representations of a group (finite group) is equal to the number of classes.

Therefore there are five irreducible representations of D_4 group.

Let the dimensions of the irreducible representations be n_1, n_2, \dots, n_5 . Then

$$\sum_{\alpha=1}^5 n_{\alpha}^2 = 8 \quad (\text{order of the group})$$

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 8$$

This can be satisfied by $n_1 = n_2 = n_3 = n_4 = 1$ and $n_5 = 2$.

Thus four irreducible representations are one dimensional and one irreducible representation is two dimensional.

Let ν_j ($j=1, 2, \dots, 5$), denote number of group elements of class C_j , and $\chi_j^{(\alpha)}$ be the characters of the class C_j . Introduce

$$\lambda_j^{(\alpha)} = \chi_j^{(\alpha)} \nu_j / n_{\alpha}$$

then $\lambda_j^{(\alpha)}$ follow the same multiplication rule as class multiplication rules.

$$\therefore \chi_2 = \pm 2$$

Using IV we get

$$\frac{1}{8} (4 + 4 + \chi_3^2 + \chi_4^2 + \chi_5^2) = 1$$

This gives $\chi_3 = \chi_4 = \chi_5 = 0$

Thus character table becomes

χ_i		Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
1	$C_1 = \{e\}$	1	1	1	1	2
1	C_2	1				-2^* (* 2 discarded)
2	C_3	1				0
2	C_4	1				0
2	C_5	1				0

Character Table Structure.

Every finite group has a one dimensional representation which assigns number 1 to each element and hence its trace $\equiv \chi_j^{(1)} = 1$ for all classes. This gives the first column of the character table.

The identity element is represented by the identity matrix and corresponding trace $= \chi_1^{(\alpha)}$ for an IRR is the dimension of the IRR. This gives the first row of the table.

The class multiplication rules are given to be

$$C_2^2 = C_1 \quad C_1 C_j = C_j, \quad j = 2, \dots, 5$$

$$C_3^2 = 2C_1 + 2C_2; C_4^2 = 2C_1 + 2C_2; C_5^2 = 2C_1 + 2C_2$$

$$C_2 C_3 = C_3; C_2 C_4 = C_5; C_2 C_5 = C_5 \quad \textcircled{\text{I}}$$

$$C_3 C_4 = 2C_5, C_3 C_5 = 2C_4, C_4 C_5 = 2C_3$$

$$v_1 = 1, v_2 = 1, v_3 = 2, v_4 = 2, v_5 = 2$$

are the number of group elements in classes C_1, \dots, C_5 .

Consider 2 dimensional IRR of D_4 . $n_\alpha = 2$

Drop index α .

$$\lambda_j = \chi_j v_j / 2$$

$$\lambda_1 = \chi_1 / 2, \lambda_2 = \chi_2 / 2, \lambda_3 = \chi_3, \lambda_4 = \chi_4, \lambda_5 = \chi_5 \quad \textcircled{\text{II}}$$

The character vectors are normalized as

$$\frac{1}{8} (|\chi_1|^2 + |\chi_2|^2 + 2|\chi_3|^2 + 2|\chi_4|^2 + 2|\chi_5|^2) = 1 \quad \textcircled{\text{IV}}$$

The class C_1 consists of $\{e\}$, the identity element only. Hence its character is equal to the dimension of the IRR

$$\therefore \chi_1 = 2 \quad \textcircled{\text{V}}$$

Next $C_2^2 = C_1 \Rightarrow \lambda_2^2 = \lambda_1$ or using (II)

$$\chi_2^2 / 4 = \chi_1 / 2 \Rightarrow \chi_2^2 = 4$$

In the character table

$\alpha \rightarrow$ column index corresponding to IRR Γ_α

$j \rightarrow$ row index corresponding to different classes g_j .

$2_j =$ number of elements in the class.

The two orthogonality theorems given

$$\sum_{\alpha} \sqrt{\frac{2_j}{2(G)}} \chi_j^{(\alpha)*} \chi_k^{(\alpha)} \sqrt{\frac{2_k}{2(G)}} = \delta_{jk}.$$

$$\text{and } \sum_j \chi_j^{(\alpha)*} \chi_j^{(\beta)} = 0 \text{ if } \alpha \neq \beta.$$

\therefore For reasons of orthogonality of Columns 1) and Col 5 (for Γ_1 and Γ_5) we must select $\chi_2 = -2$. and discard $\chi_2 = +2$.

Characters of one dimensional representations.
For a one dimensional representation Γ_α , $n_\alpha = 1$ and hence $\lambda_j = \chi_j 2_j$

Thus gives

$$\lambda_1 = \chi_1 = 1, \quad \lambda_2 = \chi_2, \quad \lambda_3 = 2\chi_3, \quad \lambda_4 = 2\chi_4 \\ \lambda_5 = 2\chi_5$$

The numbers λ_j follow the same multiplication rule as classes C_j . The class multiplication rules on p2 give $C_2^2 = C_1 \Rightarrow \lambda_2^2 = 1 \Rightarrow \lambda_2 = \pm 1$

also $4\lambda_3^2 = 2 + 2\lambda_2$

$\lambda_2 = -1$ can be seen to be inconsistent with orthogonality

$= 4 \quad (\because C_3^2 = 2C_1 + 2C_2)$

$\therefore \lambda_3^2 = 1$

$$C_2 C_3 = C_3 \Rightarrow \lambda_2 (2\lambda_3) = 2\lambda_3$$

$$\Rightarrow \lambda_2 = 1$$

~~$C_2 C_4 = C_4$~~ \Rightarrow

$$C_4^2 = 2C_1 + 2C_2 \Rightarrow \lambda_4^2 = 1$$

$$C_5^2 = 2C_1 + 2C_2 \Rightarrow \lambda_5^2 = 1$$

Orthogonality property of C_1 with the first C_1 gives

$$\lambda_1 \lambda_1 + \lambda_2 \lambda_2 + \lambda_3 \lambda_3 + \lambda_4 \lambda_4 + \lambda_5 \lambda_5 = 0$$

$$\lambda_1 + \lambda_2 + 2(\lambda_3 + \lambda_4 + \lambda_5) = 0$$

or $\because \lambda_1 = \lambda_2 = 2$ we get-

$$\lambda_3 + \lambda_4 + \lambda_5 = 0.$$

This along $\lambda_3^2 = \lambda_4^2 = \lambda_5^2 = 1$ gives:

Two of these are -1 and the third one is $+1$.

Therefore for one dimensional representations
we have three solutions. $\chi_1 = \chi_2 = 1$

$$\chi_3 = +1 \quad \chi_4 = -1 \quad \chi_5 = -1$$

$$\chi_3 = -1 \quad \chi_4 = +1 \quad \chi_5 = -1$$

$$\chi_3 = -1 \quad \chi_4 = -1 \quad \chi_5 = +1$$

and the complete character table is

		$\Gamma^{(1)}$	$\Gamma^{(2)}$	$\Gamma^{(3)}$	$\Gamma^{(4)}$	$\Gamma^{(5)}$
$\chi_1 = 1$	C_1	1	1	1	1	2
$\chi_2 = 1$	C_2	1	1	1	1	-2
$\chi_3 = 2$	C_3	1	1	-1	-1	0
$\chi_4 = 2$	C_4	1	-1	1	-1	0
$\chi_5 = 2$	C_5	1	-1	1	1	0

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