GT-02 Solved Problem
Class multiplication rules for D_4

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Abstract

This problem illustrates the construction of class multiplication rules using the information about elements of classes and group multiplication table.

Problem 1: List all the elements of symmetries of a square and construct the group multiplication table. Use the multiplication table construct the class multiplication rules.

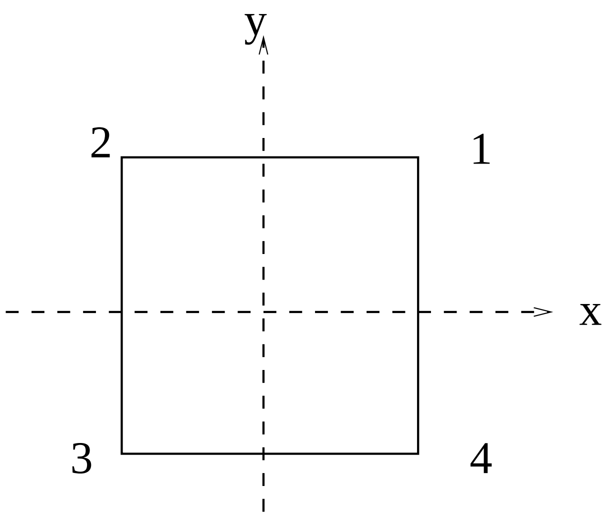


Fig. **1**

Notation: Name the group elements as follows.

Anticlockwise rotation about z- axis by angle $\pi/2 :\rightarrow 4_z$.

Anticlockwise rotation about z- axis by angle π:$\rightarrow 4_z^2$.

Anticlockwise rotation about z- axis by angle $3\pi/2$:$\rightarrow 4_z^3$.

Reflection in a plane perpendicular to X- axis $:\rightarrow m_x$

Reflection in a plane perpendicular to Y- axis $:\rightarrow m_y$

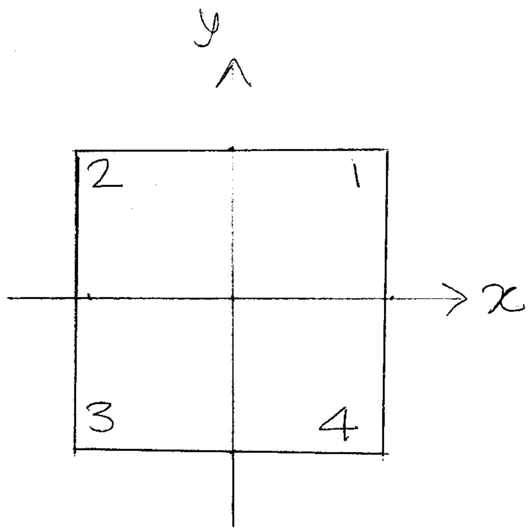
Reflection in plane containing Z- axis and diagonal 13 $:\rightarrow m_{13}$

Reflection in plane containing Z- axis and diagonal 24 $:\rightarrow m_{24}$

The classes are given to be

$$
\begin{array}{ll}
C_1 = \{e\}. & \\
C_2 = \{4_z^2\} & C_3 = \{4_z, 4_z^3\} \\
C_4 = \{m_x, m_y\} & C_5 = \{m_{13}, m_{24}\}
\end{array}
$$

Symmetries of a square



Consider a square in xy plane with its corners labeled as 1,2,3,4.

The following operations leave the square unchanged

identity operation, rotations about the z axis by angles $\pi/2$, π and $3\pi/2$. These operations will be denoted by 4_z, 4_z^2 and 4_z^3 respectively

Mirror reflection in planes perpendicular to x and y axes to be denoted by m_x and m_y.

Mirror reflections in diagonals to be called $\delta_1 \equiv m_{24}$ and $\delta_2 \equiv m_{13}$.

Therefore we have

$$D_4 = \{e, 4_z, 4_z^2, 4_z^3, m_x, m_y, m_{13}, m_{24}\}$$

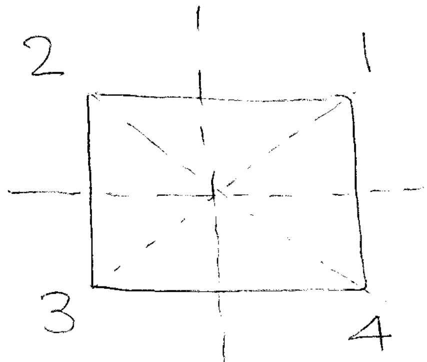
rotations $4_z, 4_z^2, 4_z^3$ are in anticlock wise direction about the z-axis.

Each group element can be represented by a permutation.

Elements of D_4 group as permutations.

The group elements will be denoted by ~~a~~ permutations

$$g = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ f_1 & f_2 & f_3 & f_4 \end{Bmatrix}$$



Here f_1, f_2, f_3 and f_4 denote the final positions of corners 1, 2, 3, 4 under group operation g. Thus we will have the following mapping on permutations.

$$e = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{Bmatrix} \qquad m_x = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{Bmatrix}$$

$$4_z = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{Bmatrix} \qquad m_y = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{Bmatrix}$$

$$4_z^2 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{Bmatrix} \qquad m_{13} = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{Bmatrix}$$

$$4_z^3 = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{Bmatrix} \qquad m_{24} = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{Bmatrix}$$

$m_x \rightarrow$ Reflection in plane $\perp$ to the x axis
YZ plane

$m_y \rightarrow$ Reflection in plane $\perp$ to y axis
XZ plane

Class multiplication

The classes of D_4 group are

$C_1 = \{e\}$ $C_2 = \{4_z^2\}$ $C_3 = \{4_z, 4_z^3\}$

$C_4 = \{m_x, m_y\}$ $C_5 = \{m_{13}, m_{24}\}$

The result of multiplying two classes C, C' is the set of all product group elements $g \in C$ and $g' \in C'$

Thus $CC' = \{gg' \mid g \in C, g \in C'\}$

$C_1^2 = \{e\}$, $C_1C_2 = C_2$ $C_1C_3 = C_3$

$C_1C_4 = C_4$ $C_1C_5 = C_5$

Next consider $C_3C_4 = \{4_z m_x, 4_z m_y, 4_z^3 m_x, 4_z^3 m_y\}$

In writing the product such as $4_z m_x$ apply m_x first, then apply 4_z.

$$4_z m_x = \underbrace{\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{Bmatrix}}_{\text{action of } 4_z} \underbrace{\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{Bmatrix}}_{\text{action of } m_x}$$

$$= \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{Bmatrix} = m_{24}$$

Here, $1 \xrightarrow{m_x} 2 \xrightarrow{4_z} 3$ $\quad 3 \xrightarrow{m_x} 4 \xrightarrow{4_z} 1$

$2 \xrightarrow{m_x} 1 \xrightarrow{4_z} 2$ $\quad 4 \xrightarrow{m_x} 3 \xrightarrow{4_z} 4$

$$4_z m_y = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{Bmatrix} \equiv m_{13}$$

because
$$1 \xrightarrow{m_y} 4 \xrightarrow{4_z} 1$$
$$2 \xrightarrow{m_y} 3 \xrightarrow{4_z} 4$$
$$3 \xrightarrow{m_y} 2 \xrightarrow{4_z} 3$$
$$4 \xrightarrow{m_y} 1 \longrightarrow 2$$

$$4_z^3 m_x = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{Bmatrix} \equiv m_{13}$$

$$4_z^3 m_y = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{Bmatrix} \equiv m_{24}$$

Therefore

$$C_3 C_4 = \{4_z m_x, 4_z m_y, 4_z^3 m_x\ 4_z^3 m_y\}$$

$$= \{m_{24}, m_{13}, m_{13}, m_{24}\}$$

$$= 2\, C_5$$

2 because each element of C_5 appears twice.

Next consider C_4^2

$$C_4^2 = \{ m_x^2, m_y^2, m_x m_y, m_y m_x \}$$

$$m_x^2 = e \quad m_y^2 = e$$

$$m_x m_y = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{Bmatrix}$$

$$= 4_z^3$$

$$m_y m_x = \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{Bmatrix}$$

$$= 4_z^2$$

$$C_4^2 = \{ e, e, 4_z^2, 4_z^2 \}$$

$$= 2C_1 + 2C_2$$

The class multiplication rules for D_4 are

$C_2^2 = C_1$ $\quad$ $C_2 C_3 = C_3$ $\quad$ $C_2 C_4 = C_4, C_2 C_5 = C_5$

$C_3^2 = 2C_1 + 2C_2$ $\quad$ $C_3 C_4 = 2C_5$ $\quad$ $C_3 C_5 = 2C_4$

$C_4^2 = 2C_1 + 2C_2$ $\quad$ $C_4 C_5 = 2C_3$

$C_5^2 = 2C_1 + 2C_2$