QM-13 Simply master piece

Potential Problems in One Dimension

Existence of bound states in One Dimension

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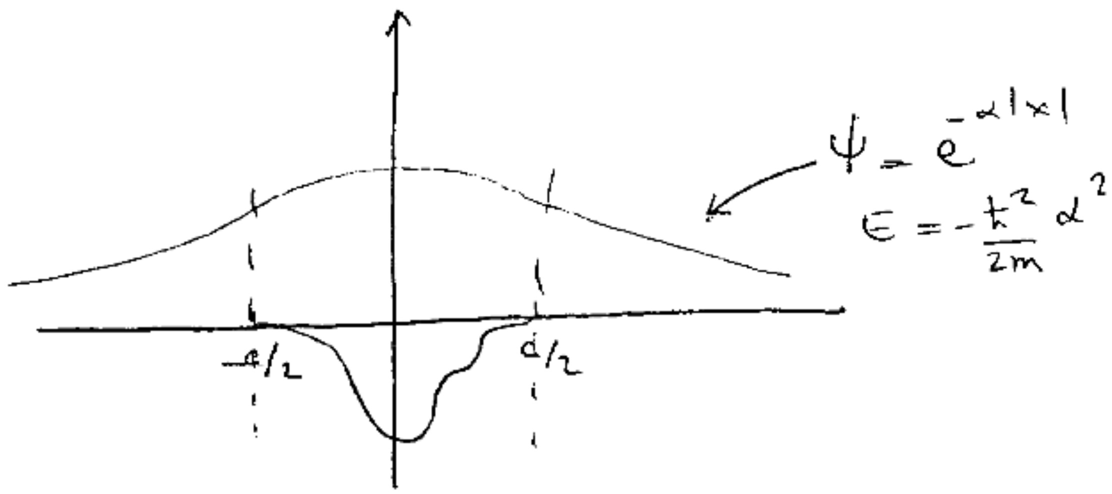
Abstract

An attractive potential always has a bound state in one dimension. What is an attractive potential? For an answer and a nice argument for the statement read on this master piece from C. G. Callan Jr. You are asked to resolve a contradiction with an uncertainty principle argument.

1. One-Dimensional Odds and Ends

The Schrödinger equation may be solved exactly for certain classes of potential in one dimension. This gives us the ability to provide *analytic* treatments of basic topics such as energy levels and scattering amplitudes. The examples are unrealistic but instructive. We start with some discussion of energy levels.

You have probably heard that, in one dimension, an *arbitrarily* weak attractive potential has at least one bound state. We will begin with a rough and ready argument for this proposition which must work in the limit as the potential scales to zero. In the figure, we plot a notional weak potential with support in $[-a/2, a/2]$ together with a sketch of a weakly bound bound state wave function.



Assuming small binding energy, the variation of the wave function across the region of the potential can be neglected. Integrating the Schrödinger equationacross that region gives

$$-2\alpha = (\psi'(a/2) - \psi'(-a/2))/\psi(0) = \frac{2m}{\hbar^2}\int V(x)dx$$

This fixes α, and therefore E, and only makes sense if the integral of the potential is negative. The resulting expression for the energy is

$$E_{b.s.} \simeq -\frac{\hbar^2}{2m}\left[\frac{m}{\hbar^2}\int dxV\right]^2 = -\frac{m}{2\hbar^2}\left[\int dx \cdot V(x)\right]^2 \ .$$

So, an attractive potential (its integral is negative), no matter how weak, has at least one bound state. We have identified the energy and wave function in the case that the potential is very weak.

For the purposes of this argument, all that mattered was where the potential was centered and the value of its integral. We would have gotten essentially the same state and energy if we had replaced the complicated potential by any potential with the same integral. This leads us to contemplate a special and instructive class

Question For You: This brings us to the following question:

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Consider a particle in one dimensional square well potential

$$V(x) = \begin{cases} 0 & 0 \le x \le L \\ V_0 & \text{otherwise} \end{cases} \quad (1)$$

For a bound state we should have average energy less than V_0.
$\langle E \rangle = \langle T \rangle + \langle V \rangle < V_0$
where $\langle T \rangle$ and $\langle V \rangle$ are the averages of the kinetic and the potential energies, respectively. If the particle is to be confined to a region of size L, use the uncertainty principle to get a rough estimate of average kinetic energy, $\langle T \rangle$. This argument gives an approximate minimum value of $V_0 a^2$ required for a bound state to exist.

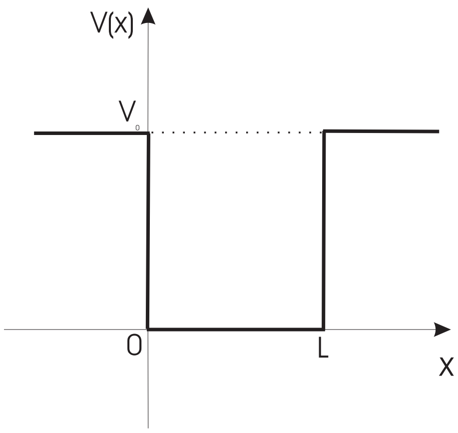

Fig. **1** sqrwel1

However, it is known that for an attractive potential in one dimension a bound state always exists. Resolve the contradiction between the uncertainty principle and the argument for existence of a bound state on one dimension for an attractive potential.