QM-09 Lecture Notes
Pictures in quantum mechanics
The Interaction Picture*

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Abstract

The interaction picture, also known as Dirac picture, or the intermediate picture, is defined by splitting the Hamiltonian in two parts, the free and the interaction parts. In interaction picture equation of motion for the observables is free particle equation. The state vector satisfies Schrodinger equation with interaction Hamiltonian giving the rate of time evolution.

§1 Pictures in quantum mechanics

We recall that the time evolution of quantum states is given by the time dependent Schrödinger equation.

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle. \tag{1}$$

and the operators do not carry any time dependence. The dynamical variables, represented by operators are specified once for all and do not evolve with time. This 'picture' of time evolution is called Schrodinger picture.

Recall that the state vector by itself is not measurable quantity. Only the average values and the probabilities are measurable quantities. This allows the possibility of describing the time dependence in several ways. Heisenberg [1] and Dirac pictures are two important alternate descriptions of time evolution.

§2 Interaction picture

We shall now discuss the interaction picture, also known as Dirac picture. We shall denoting the Schrodinger picture kets and operators by $|\psi\rangle_S, X_S$ etc. and $|\psi\rangle_I, X_I$ etc will denote the corresponding quantities in the interaction picture.

Let the Hamiltonian of the system be written as sum of two parts

$$H = H_0 + H'. \tag{2}$$

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H_0, H' will be called free part and the interaction part of the Hamiltonian H, respectively. While H_0 is assumed to be independent of time, the interaction Hamiltonian may or may not depend on time. The state of a system in the interaction picture are defined by

$$|\psi t\rangle_I = e^{iH_0t/\hbar}|\psi t\rangle_S. \tag{3}$$

and the dynamical variables of the interaction picture are defined by demanding that the average values in the interaction and Schrödinger pictures coincide at all times:

$$_I\langle\psi|X_I|\psi\rangle_I \equiv_S \langle\psi|X_I|\psi\rangle_S. \tag{4}$$

Substituting

$$|\psi t\rangle_S = e^{-iH_0t/\hbar}|\psi t\rangle_I, \tag{5}$$

from (3) we get

$$_I\langle\psi|X_I|\psi\rangle_I = {}_I\langle\psi|e^{iH_0t/\hbar}X_I e^{-iH_0t/\hbar}|\psi\rangle_I. \tag{6}$$

Therefore, we use

$$X_I = e^{iH_0t/\hbar}X_S e^{-iH_0t/\hbar} \tag{7}$$

to define the an interaction picture dynamical variables.

The time dependence of the interaction picture operators is very simple and is governed by the free Hamiltonian H_0:

$$i\hbar\frac{dX_I(t)}{dt} = \big[X_I, H_0\big]. \tag{8}$$

As an example, it should be obvious that, the free particle Hamiltonian H_0 in the interaction picture remains identical with H_0:

$$(H_0)_I = e^{iH_0t/\hbar}H_0 e^{-iH_0t/\hbar} = H_0. \tag{9}$$

On the other hand, even though the interaction part of the Hamiltonian, H', may be independent of time, the interaction picture Hamiltoinan H'_I

$$H'_I(t) = e^{iH_0t/\hbar}H'e^{-iH_0t/\hbar} \tag{10}$$

is different from H' and depends explicitly on time. The time dependence of the state vector in the interaction picture is governed by this operator H'_I, the interaction part of Hamiltonian,H', transformed to the interaction picture. We will now derive the differential equation which gives the evolution of the state vectors. For this purpose we begin with (3)

$$\begin{aligned}
i\hbar\frac{d}{dt}|\psi t\rangle_I &= i\hbar\frac{d}{dt}\Big(e^{iH_0t/\hbar}|\psi t\rangle_S\Big) \\
&= i\hbar\left(\frac{d}{dt}e^{iH_0t/\hbar}\right)|\psi t\rangle_S + e^{iH_0t/\hbar}\Big(i\hbar\frac{d}{dt}|\psi t\rangle_S\Big) + \qquad (11) \\
&= -H_0e^{iH_0t/\hbar}|\psi_0\rangle_S + e^{iH_0t/\hbar}\big(H_0 + H'\big)|\psi t\rangle_S \\
&= e^{iH_0t/\hbar}(-H_0)|\psi_0\rangle_S + e^{iH_0t/\hbar}\big(H_0 + H'\big)|\psi t\rangle_S \\
&= e^{iH_0t/\hbar}\big(H'\big)|\psi t\rangle_S \qquad (12)
\end{aligned}$$

Next, we need to express the Schrödinger picture state vector,$|psit\rangle_S$, in the right hand side in terms of the interaction picture state vector $|\psi t\rangle_I$. Thus we get

$$i\hbar\frac{d}{dt}|\psi t\rangle_I \quad = \quad e^{iH_0t/\hbar}\big(H'\big)e^{-iH_0t/\hbar}|\psi t\rangle_I. \tag{13}$$

Thus we get the desired equation for time evolution of the state vectors in the interaction picture in the final form

$$\boxed{i\hbar\frac{d}{dt}|\psi t\rangle_I = H'_I|\psi t\rangle_I} \tag{14}$$

where H'_I is given by Eq.(10). The state vector evolves with the interaction part of the Hamiltonian.

References

[1] A. K. Kapoor, *Heisenberg Picture of Quantum Mechanics*, Lecture Notes :: qm-lec-09006.pdf