CM-05 Lecture Notes
Motion in Spherically Symmetric Potentials*

http://0space.org/node/898

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Contents

§1 Cyclic Coordinates

Newton's EOM and also the Euler Lagrange equations, are second order differential equations. The knowledge of conservation laws and of constants of motion greatly simplifies the task of obtaining solutions to the EOM. This is most clearly seen in one dimension, where use of conservation law for energy reduces the problem to quadrature.

The existence of conservation laws can be inferred from the symmetry properties of the Lagrangian by appealing to Noether's Theorem. However, this requires some amount of training in most cases. There is a class of symmetries and associated conservation laws, which can be can be arrived at by inspection. If the Lagrangian is independent of a

* KApoor//CM-Lecs-05.tex; Ver 16.x June 22, 2016

particular coordinate q_r, it depends only on generalised velocity $\dot{q}_r$ we have a conservation law. Such a coordinate is called **cyclic coordinate or ignorable coordinate.** If q_r is a Cyclic coordinate, then

$$\frac{\partial L}{\partial q_r} = 0 \tag{1}$$

and the Euler's Lagrange EOM

$$\frac{\partial L}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = 0 \tag{2}$$

implies that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} = 0 \Longrightarrow \qquad \frac{\partial L}{\partial \dot{q}_r} = \text{constant}. \tag{3}$$

Hence the canonical momentum conjugate to q_r is a constant of motion.

The expression for of velocity corresponding to the cyclic coordinates can be obtained from Eq.(3) as a function of other coordinates and velocities. Thus the cyclic coordinate and the corresponding velocity then get eliminated from the equations of motion. In fact one can write down a Lagrangian for the remaining coordinates, which gives correct EOM for them. We now give this process.

Let $q_a, a = 1, 2, 3, \ldots, m$ be cyclic coordinates and corresponding momenta $p_a = \beta_a$ are constants, say β_a. Then

$$p_a(q, \dot{q}, t) = \beta_a \tag{4}$$

We use (4) to solve for velocities $\dot{q}_a, a = 1, 2, \ldots, m$ and express them as functions of the constants β_a and the remaining coordinates q_k and velocities $\dot{q}_k$

$$\dot{q}_a = \dot{q}_a(\beta, q_k, \dot{q}_k, t) \tag{5}$$

The function

$$R = L - \sum_{a=1}^{m} \dot{q}_a \left(\frac{\partial L}{\partial \dot{q}_a}\right) \tag{6}$$

$$= L - \sum_{a=1}^{m} \beta_a \dot{q}_a \tag{7}$$

when expressed in terms of $\dot{q}_k, q_k, \beta_a$, describes the dynamics of the remaining coordinates.

Note that the new 'Lagrangian' cannot be obtained from the old one by simply eliminating the velocities $\dot{q}_a$ by making use of (4).

§2 Spherically Symmetric Potential

§2.1 Reduction of Two Body Problem

Consider two particles interacting through a potential. Which only on the distance between the two. The Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - V(|\vec{r}_2 - \vec{r}_1|). \tag{8}$$

We now introduce the position vector of the centre of mass $\vec{R}$, and the relative coordinate $\vec{r}$ by means of

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{(m_1 + m_2)}, \qquad \vec{r} = (\vec{r_1} - \vec{r_2}), \tag{9}$$

and express the Lagrangian in terms of the new variables takes the form

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r). \tag{10}$$

where M is the total mass, and μ is reduced mass.

$$M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{11}$$

The equations of motion for $\vec{R}$ are simple and just state that the center of mass moves with constant velocity. The rest of the terms in the Lagrangian describe motion of a particle of mass μ in a potential $V(r)$. The potential $V(r)$ depends only on $r = |\vec{r}|$ and is independent of θ, ϕ. Such potentials are called spherically symmetric potentials.

§2.2 Conservation Laws

Let the Lagrangian of a system to be studied be given by

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \tag{12}$$

The Lagrangian does not contain time explicitly, hence we obtain energy conservation

$$\frac{1}{2}\mu\dot{\vec{r}}^2 + V(r) = E \qquad \text{(constant)}. \tag{13}$$

The Lagrangian is also invariant under rotations about any axis and in particular about the coordinate axes. This gives us conservation of angular momentum. Thus we have

$$\vec{L} = \mu\vec{r} \times \vec{v} = \text{constant of motion} \tag{14}$$

§2.3 Reduction of solution to quadratures

We shall now make use of the conservation laws to give solution of motion in a spherically symmetric potential to quadratures.

Since $\vec{L} = \mu\vec{r} \times \vec{v}$ is a constant of motion the magnitude as the direction of $\vec{L}$ does not change with time. Also $\vec{r}$ and $\vec{v}$ always perpendicular to $\vec{L}$ which points in a fixed direction. Hence $\vec{r}$ and $\vec{v}$ remain in the plane perpendicular to $\vec{L}$. Therefore for a particle in a spherically symmetric potential, the motion is confined to a plane.

If $\vec{L}$ is zero, then $\vec{r} \times \vec{v} = 0$ and $\vec{r}$ will always be parallel to $\vec{v}$ and the particle moves in a straight line.

We, therefore, start with Lagrangian for a particle in two dimensions in plane polar coordinates

$$\mathcal{L} = \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - V(r). \tag{15}$$

The expression for energy, associated with relative motion, given by

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + V(r). \tag{16}$$

Since $\dot{\phi}$ is a cyclic coordinate we have

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi} = \text{constant, say} L. \tag{17}$$

$\mu r^2 \dot{\phi}$ is in fact seen to be equal to the magnitude of angular momentum. The velocity $\dot{\phi}$ can be eliminated using

$$\dot{\phi} = \frac{L}{2\mu r^2} \tag{18}$$

Making use of (16) and (18) we get,

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = \frac{1}{2}\mu \dot{r}^2 + V_{\text{eff}}(r) \tag{19}$$

where we have introduced the notation

$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2\mu r^2}. \tag{20}$$

The radial motion of the particle is seen to be like motion in one dimension with $V_{eff}(r)$ as the potential. Solving (20) for $\dot{r}$ we get

$$\dot{r}^2 = \frac{2}{\mu}\left(E - V(r) - \frac{L^2}{2\mu r^2}\right), \tag{21}$$

or

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - V(r)) - \frac{L^2}{2mr^2}}. \tag{22}$$

Integrating we get

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}(E - V(r)) - \frac{L^2}{2mr^2}}} + c \tag{23}$$

This equation when inverted gives r as a function of t and using (18) we get ϕ as a function of time

$$\dot{\phi} = \frac{L}{mr^2} \tag{24}$$

$$d\phi = \frac{L}{mr^2} dt \tag{25}$$

which gives

$$\phi = \int \frac{L}{mr^2} dt + \text{constant} \tag{26}$$

It is understood that r is to be obtained a function of time from (23) and is substituted in the R.H.S of (26).

The equation of the orbit, relation between r and ϕ, is easily obtained from Eq.(22) and Eq.(24)

$$\frac{dr}{d\phi} = \left(\frac{dr}{dt}\right) \Big/ \left(\frac{d\phi}{dt}\right), \tag{27}$$

$$= \sqrt{\frac{2}{\mu}\left(E - V(r) - \frac{L^2}{2mr^2}\right)} \times \left(\frac{\mu r^2}{L}\right). \tag{28}$$

Therefore, we get

$$dr = \sqrt{\frac{2}{\mu}\left(E - V(r) - \frac{L^2}{2mr^2}\right)} \times \left(\frac{\mu r^2}{L}\right) d\phi, \tag{29}$$

or

$$\phi = \int \left(\frac{L}{\mu r^2}\right) \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V(r) - \frac{L^2}{2\mu r^2}\right)}} + \text{constant}. \tag{30}$$

Thus the problem of determining r and ϕ as a function of time and the problem of finding the orbit has been reduced to quadratures. In many cases it turns out to be useful to change variable to $u = 1/r$, and write Eq.(30) can be rewritten as

$$\phi = \int \frac{1}{r^2} \frac{dr}{\sqrt{\left(\frac{2\mu E}{L^2} - \frac{2\mu V(r)}{L^2} - \frac{1}{r^2}\right)}} + \text{constant}, \tag{31}$$

or

$$\phi - \phi_0 = -\int_{u_0}^{u} \frac{du}{\sqrt{\left(\frac{2\mu E}{L^2} - \frac{2\mu}{L^2} V(\frac{1}{u}) - u^2\right)}}. \tag{32}$$

§3 Differential Equations of the Orbit

The equation of motion for a particle in a spherical symmetric potential can be solved making use of the conservation laws. Here we obtain a second order differential equation, which is sometimes easy to solve.

The Lagrangian for a particle in spherical symmetric potential, assuming motion in a plane, is

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - V(r) \tag{33}$$

The Euler Lagrangian equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \tag{34}$$

$$\frac{d}{dt}(\mu \dot{r}) - \mu r \dot{\phi}^2 + \frac{\partial V}{\partial r} = 0 \tag{35}$$

and

$$\mu r^2 \dot{\phi} = \text{constant, say} L \tag{36}$$

Let $f(r) = -\frac{\partial V}{\partial r}$ be the force law, then (35) and (36) give

$$\mu \ddot{r} = f(r) + \frac{L^2}{\mu r^3} \tag{37}$$

Next use Eq.(36) to convert $\ddot{r}$ into $\frac{d^2 r}{d\phi^2}$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\left(\frac{d\phi}{dt}\right) \tag{38}$$

$$= \frac{L}{\mu r^2}\frac{dr}{d\phi} \tag{39}$$

and

$$\frac{d^2r}{dt^2} = \frac{L}{\mu r^2}\frac{d}{d\phi}\frac{L}{\mu r^2}\frac{dr}{d\phi} = \frac{L^2}{\mu^2 r^2}\frac{d}{d\phi}\frac{1}{r^2}\frac{dr}{d\phi} \tag{40}$$

Changing variables from r to $u = \frac{1}{r}$, we get

$$\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \tag{41}$$

and

$$\frac{d^2u}{d\phi^2} = -\frac{d}{d\phi}\frac{1}{r^2}\frac{dr}{d\phi^2}. \tag{42}$$

Using (40) and (42) in Eq.(37), we get

$$\frac{L^2}{\mu r^2}\frac{d}{d\phi}\frac{1}{r^2}\frac{dr}{d\phi} = f(r) + \frac{L^2}{\mu r^3} \tag{43}$$

$$\therefore \quad -\frac{L^2}{r^2}\frac{1}{\mu}\frac{d^2u}{d\phi^2} = f\left(\frac{1}{u}\right) + \frac{L^2}{\mu}u^3 \tag{44}$$

or

$$\frac{L^2u^2}{\mu}\frac{d^2u}{d\phi^2} = -f\left(\frac{1}{u}\right) - \frac{L^2}{\mu}u^3 \tag{45}$$

This is the required differential equation of the orbit. If the equation of the orbit is known the force law can be found.

§3.1 The Kepler Problem

We shall now solve the differential Eq.(45) for the attractive inverse square law of force.

$$f = -\frac{k}{r^2} = -ku^2 \tag{46}$$

The differential equation of the orbit, Eq.(45) becomes

$$\frac{L^2}{\mu}\frac{d^2u}{d\phi^2} = k - \frac{L^2}{\mu}u. \tag{47}$$

or

$$\frac{d^2u}{d\phi^2} + u - \frac{\mu k}{L^2} = 0 \tag{48}$$

We now change the variable to $w = u - \frac{mk}{L^2}$ to get

$$\frac{d^2w}{d\phi^2} + w = 0 \tag{49}$$

This equation can be solved immediately to give

$$w = b\cos(\phi - \phi_0) \tag{50}$$

or

$$u = \frac{\mu k}{L^2} + b\cos(\phi - \phi_0) \tag{51}$$

Hence the equation of the orbit can be written in the form

$$\frac{1}{r} = \frac{\mu k}{L^2}(1 + \epsilon \cos(\phi - \phi_0)) \tag{52}$$

Recall $L =$ angular momentum, ϵ, and ϕ_0 are constants of motion. We shall now relate r_1, r_2, ϵ and energy E.

If r_1 and r_2 are the minimum and maximum values of r, then from (19)

$$\frac{1}{r_{1,2}} = \frac{\mu k}{L^2}(1 \pm \epsilon) \tag{53}$$

We relate these values $r_{1,2}$ in terms to E. At maximum of r and also at minimum, $\dot{r} = 0$ and the energy conservation equation

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) \tag{54}$$

becomes

$$E = \frac{L^2}{2\mu r^2} - \frac{k}{r} \qquad (\because \dot{r} = 0), \tag{55}$$

or

$$\frac{1}{r^2} - \frac{2\mu k}{L^2 r} - \frac{2\mu E}{L^2} = 0. \tag{56}$$

The values r_1 and r_2 are roots of this equation, hence

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2\mu u}{L^2} \tag{57}$$

$$\frac{1}{r_1}\frac{1}{r_2} = \frac{-2\mu E}{L^2} \tag{58}$$

Eq.(53) and (58) give us the desired expressions for ϵ in terms of energy

$$\frac{\mu^2 k^2}{L^4}(1 - \epsilon^2) = \frac{-2\mu E}{L^2} \tag{59}$$

or

$$\epsilon^2 - 1 = \frac{2EL^2}{\mu k^2} \tag{60}$$

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}} \tag{61}$$

Therefore, the final form of the equation of the orbit is

$$\frac{1}{r} = \frac{\mu k}{L^2}(1 + \epsilon \cos(\phi - \phi_0)) \tag{62}$$

ϕ_0 is the value of ϕ at the turning point $r = r_{min}$. This equation represents a conic section with eccentricity $= \epsilon$. Types of orbits traversed, corresponding to different values of ϵ, are summarised below.

$\epsilon < 1$	Ellipse	if $E < 0$	bounded motion
$\epsilon = 0$	Circle	$E = -\frac{\mu k^2}{2L^2}$	bounded motion
$\epsilon = 1$	Parabola	$E = 0$	unbounded motion
$\epsilon > 1$	Hyperbola	$E > 0$	unbounded motion

§3.2 Kepler's Laws

1. The motion in $\frac{1}{r}$ gravitational fields is in elliptic orbits.

2. Areal velocity is constant follows from the conservation laws of angular momentum.

$$\mu r^2\dot{\phi} = \text{constant, say} L \Longrightarrow r^2\dot{\phi} = \frac{L}{\mu} = \text{constant}$$

$$\text{Areal velocity} = \lim_{\Delta t\to 0} \frac{\text{area swept in time}\Delta t}{\Delta t} \quad (63)$$

$$= \lim_{\Delta t\to 0} \frac{1}{2}r^2\frac{\Delta\phi}{\Delta t} = \frac{1}{2}r^2\dot{\phi} \quad (64)$$

$$= \frac{L}{2\mu} = \text{constant} \quad (65)$$

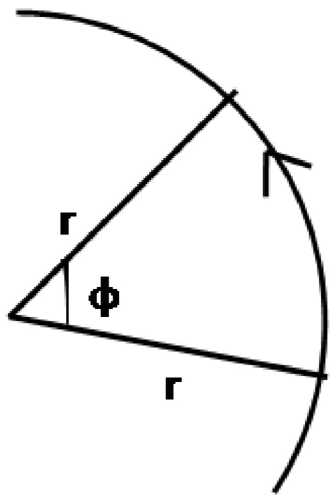

Fig. **1**

3. To calculate the time period and verify Kepler's third law, since areal velocity is constant

$$\text{Time Period } T = \frac{\text{Total area of the orbit}}{\text{areal velocity}} \quad (66)$$

$$= \frac{\pi ab}{(\frac{L}{2\mu})} = \frac{2\mu}{L}(\pi ab) \quad (67)$$

Now semi major axis

$$2a = r_1 + r_2 = \frac{k}{|E|} \quad (68)$$

where (24) and (25) has been used. Remember E is negative for elliptic orbits, so $E = -|E|$. Now

$$b = a\sqrt{1-\epsilon^2} = a\left(\frac{2|E|L^2}{\mu k^2}\right)^{\frac{1}{2}}, \tag{69}$$

Using Eq.(67)-(69) we get

$$T = \pi ab\frac{2\mu}{L} = \frac{2\pi a\mu}{L}a\left(\frac{2|E|L^2}{\mu k^2}\right)^{\frac{1}{2}} \tag{70}$$

use $|E| = \frac{k}{2a}$, from (68), and eliminate E

$$T = \frac{2\pi a^2\mu}{L}\left(\frac{2L^2}{\mu k^2}\frac{k}{2a}\right)^{\frac{1}{2}} = \frac{2\pi a^2\mu}{L}\left(\frac{L^2}{k\mu a}\right)^{\frac{1}{2}}, \tag{71}$$

$$= 2\pi a^{\frac{3}{2}}\sqrt{\frac{\mu}{k}}, \tag{72}$$

Now use $k = GMm$ and $\mu = \frac{mM}{M+m}$ to get

$$\boxed{T^2 = \frac{4\pi^2 a^3}{G(M+m)}.} \tag{73}$$

In this expression mass m of the planet can be neglected compared to the mass of the sun M. This proves Kepler's law that square of time period is proportional to the cube of semi major axis. Note that the time period has a tiny dependence on the mass of planet which has been neglected.

§3.3 Hyperbolic Orbits

We will now derive equation of orbit when $E > 0$ and the orbits are hyperbolic. The distance of closest approach corresponds to $r =$ minimum and $\frac{1}{r}$ will be a maximum. This happens at $\cos(\phi - \phi_0) = 1$

$$\tfrac{1}{r} = a(1 + \epsilon\cos(\phi - \phi_0))\ r = r_{min} \Longrightarrow \phi = \phi_0$$

As $r \longrightarrow \infty, (note\ \epsilon > 0)\frac{1}{r} \longrightarrow 0\ and \cos(\phi - \phi_0) \longrightarrow \frac{-1}{\epsilon}$

$$\phi - \phi_0 = \pi \pm \cos^{-1}\left(\frac{1}{\epsilon}\right) \tag{74}$$

$$\phi_\pm = \phi_0 + \pi \pm \cos^{-1}\left(\frac{1}{\epsilon}\right). \tag{75}$$

The total deflection of the particle in hyperbolic orbit is

$$\theta = \phi_+ - \phi_- = 2\cos^{-1}\left(\frac{1}{\epsilon}\right) \tag{76}$$

or

$$\cos\left(\frac{\theta}{2}\right) = \frac{1}{\epsilon} = \left(1 + \frac{2EL^2}{\mu k^2}\right)^{-1} \tag{77}$$

$$\cos\left(\frac{\theta}{2}\right) = \left(1 + \frac{2EL^2}{\mu k^2}\right)^{-1} \tag{78}$$

$$= \frac{\mu k^2}{(\mu k^2 + 2EL^2)} \tag{79}$$

§3.4 Runge Lenz vector

For the Kepler problem there is an additional constant of motion, given by

$$\vec{N} = \vec{v} \times \vec{L} + \frac{k\vec{r}}{r} \tag{80}$$

Proof: The equations of motion are

$$\frac{d}{dt} m\vec{v} = -\vec{\bigtriangledown}\left(\frac{k}{r}\right) = +\frac{k\vec{r}}{r^3} \tag{81}$$

$$\frac{d\vec{N}}{dt} = \frac{d\vec{v}}{dt} \times \vec{L} + \vec{v} \times \left(\frac{d\vec{L}}{dt}\right) + \frac{k}{r}\frac{d\vec{r}}{dt} - k\vec{r}\left(\frac{\dot{r}}{r^2}\right) \tag{82}$$

$$= \frac{1}{\not{m}}\left(\frac{k\vec{r}}{r^3}\right) \times (\vec{r} \times \not{m}\vec{v}) + \frac{k\dot{\vec{r}}}{r} - k\vec{r}\left(\frac{\vec{r}\cdot\vec{v}}{r^3}\right) \tag{83}$$

$$= \frac{-1}{\not{m}}\frac{k\,\vec{r}^{\,2}}{r^3}\not{m}\vec{v} + \frac{k(\vec{r}.\vec{v})\vec{r}}{r^3} + \frac{k\vec{v}}{r} - k\vec{r}\frac{(\vec{r}\cdot\vec{v})}{r^3} \tag{84}$$

$$= 0 \tag{85}$$

§4 General Properties of Motion

The radial motion in three dimensions in a spherically symmetric potential is just like motion in one dimension. The following discussion of motion in one dimension can be usefully extended to the radial motion in a spherically symmetric potential, if we replace the potential by the effective potential.

$$V(x) \longrightarrow V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2} \tag{86}$$

§4.1 Motion in one dimension

We first recall a few general properties of motion of particle in one dimension. These will be used to discuss motion in a spherically symmetric potential

Equilibrium points

If a particle, moving in one dimension, is released from rest at some point, it will in general move towards lower potential energy. If it is released at a minimum or maximum of the potential, it will remain at rest It therefore follows that at these points $\dot{x} = \ddot{x} = 0$.

Let x_0 be a point where the particle in equilibrium.

If the point x_0 is a minimum of the potential and the particle is disturbed slightly, it will execute oscillations about the minimum. In this case we say that the point x_0 is a point of {it stable equilibrium}.

If the equilibrium point is a maximum of the potential and even a slightest disturbance will make the particle move away from the equilibrium. In this case we say that the *equilibrium is unstable.*

Turning points

1. A particle moving in a potential cannot go to regions where its energy is less than the potential energy. Its motion is confined to those values of x where

$$V(x) \leq E$$

. To see this note that we must have

$$E = \frac{p^2}{2m} + V(x) \tag{87}$$

$$\therefore E \geq V(x) \tag{88}$$

$$\because \text{ K.E.} = \frac{p^2}{2m} > 0 .$$

2. The region, the set of values of x where (87) holds, is called *classically accessible region.*

3. The points where $E = V(x)$ are called *turning points.* At a turning point the velocity becomes zero, $\dot{x} = 0$.

Properties of motion in three dimensions can be obtained by using the effective potential and using similar arguments.

Range of energies for bounded motion

Assuming a continuous potential $V(x)$, the potential will have a minimum or a maximum between two turning points. For a given energy, a particle will execute a bounded motion if the potential has two turning points such that it has a minimum between the turning points.

§4.2 Motion in spherically symmetric potential

It will be useful to recall that for a motion in three dimensional spherically symmetric potential

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \tag{89}$$

The similarity of the above equation with Eq.(87) should be noted.

Nature of Orbits

1. Conservation of angular momentum implies the motion of particle is in a plane
2. If angular momentum is zero, $\vec{r} \times \vec{p} = 0. \implies \vec{r}$ and $\vec{p}$ are parallel. In this case the particle moves in a straight line.
3. If angular momentum is nonzero, the bounded or unbounded nature of the orbit can be decided in the same fashion as in one dimension by looking at the plot of the *effective potential*.
4. The motion is always confined to region where

$$E \geq V_{\rm eff}(r). \tag{90}$$

For a bounded motion, r varies between two extreme values r_1 and r_2, which correspond to the turning points. For these points the radial velocity becomes zero and the total energy is given by $V_{\rm eff}(r_1) = V_{\rm eff}(r_2) = E$.

Circular orbits

1. The equilibrium in one dimension correspond to a fixed value of $x(t) =$ constant and $\dot{x} = \ddot{x} = 0$ for all times. In three dimensions $r(t) = \text{constant}, R$ corresponds to a circular orbit. For a circular orbit of radius R, we will have $\dot{r} = \ddot{r} = 0$ for all times. Thus the radius of a circular orbit is given by minima and maxima of the effective potential, (use (89)).

$$\frac{dV_{\rm eff}(r)}{dr}\Big|_{r=R} = 0. \tag{91}$$

2. $r =$ constant. and angular momentum conservation $mr^2\dot{\phi} =$ constant imply

$$\dot{\phi} = \text{constant}.$$

Thus the particle moving in a circular orbit has a constant constant angular velocity.

3. For a bounded motion we will have $r_1 < r < r_2$ and for a circular orbit of radius R, we must have $r_1 = r_2 = R$ and the energy is given by $E = V_{\rm eff}(R)$. See Fig.2.

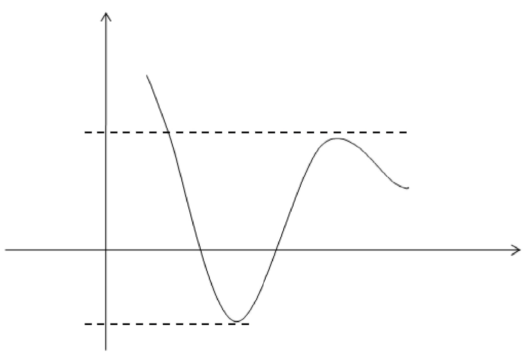

Fig. **2**

Stability of circular orbits

For a circular orbit radial velocity and acceleration are zero. Hence $m\ddot{r} = 0 \Longrightarrow \frac{\partial}{\partial r}\left(V(r) + \frac{L^2}{mr^2}\right) = 0$. Thus the circular orbits correspond to the maxima and minima of the effective potential $V_{\text{eff}}(r)$.

The maximum corresponds to an unstable circular orbit orbit. The minimum corresponds to a stable circular orbit.

$$r = r_0 + \eta$$

Fall to center

Let V(r) be finite as r $\longrightarrow$ 0. Then $V_{\text{eff}}(r) \longrightarrow \infty$ as $r \longrightarrow 0$ and a particle cannot reach r=0 for any value of E. However, for certain singular potentials the particle can reach center. Consider, for example, the case of a potential $V = \frac{-g}{r^4}, \quad g > 0$. Then the effective potential is

$$V_{\text{eff}} = \frac{-g}{r^4} + \frac{L^2}{2mr^2} \tag{92}$$

A sketch of the effective potential is shown in Fig.3. If $E >$ maximum of V_{eff}, then a particle coming from large distance can fall to center.

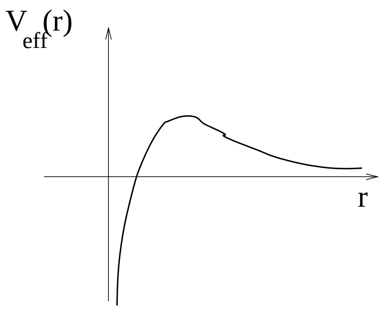

Fig. **3** Fall to centre

Escape to ∞

Assume $V(r) \longrightarrow 0 \; as \; r \longrightarrow \infty$

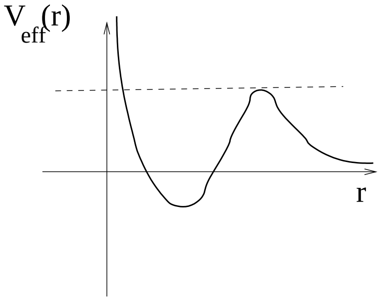

Fig. **4** Escape to infinity

A particle moving out can escape to infinity if $E >$ maximum of V_{eff} for $r > 0$.