CM-3 Lecture Notes
Action Principle *

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§1 Hamilton's principle

§1.1 Configuration space

Let us consider a system with N degrees of freedom. At a given time the system is completely specified by giving values the values of N generalised coordinates $q_1(t), \ldots, q_1(N)$. We may arrange $q's$ in a row to form an N component vector

$$q = (q_1(t), \ldots, q_N(t)) \tag{1}$$

the N component vector can be represented by points in an N dimensional space called configuration space. Conversely a point in configuration space

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represents a possible set of values of $(q_1, \ldots, q_N)$

We say that possible states of system are given by points in configuration space.

With time q' change and also does the position of the point representing the system. Thus with time, the point representing the system will trace out a path in configuration space. As Euler Lagrange equations are second order differential equations, the motion of the system, the state at any time is completely known if we specify the initial values of q and $\dot{q}$ at some time t_0.. With this as input solving the Euler Lagrange equations of motion give the generalised coordinates $q(t)$ at all times hence the path followed by the system in the configuration space is known.

An equivalent way of specifying the motion completely is to give the coordinates q at two different times t_1 and t_2. Thus we are looking for solution of Euler Lagrange equations.

$$q(t) = (q_1(t), \ldots, q_N(t)) \quad \text{for} \quad t_1 \leq t \leq t_2 \tag{2}$$

when their values at the initial time and final time

$$q(t_1) = (q_1(t_1), \ldots, q_N(t_1)), \qquad q(t_2) = (q_1(t_2), \ldots, q_N(t_2)) \tag{3}$$

are known. This amounts to asking what path is followed in configuration space, if we know the end points P_1 and P_2. Several paths in configuration space with fixed end points are shown below.

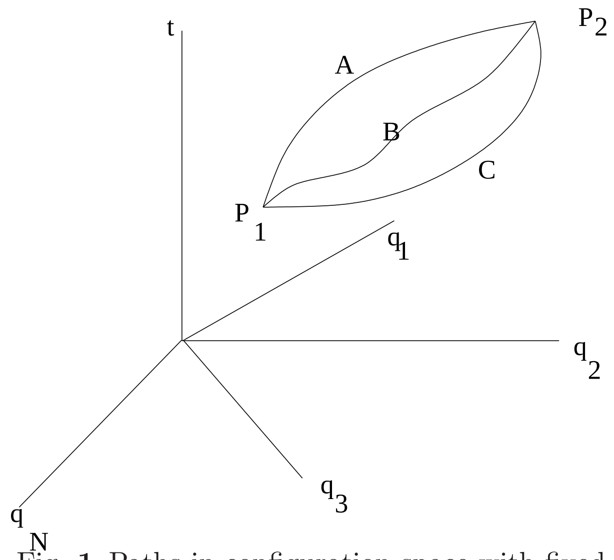

Fig. **1** Paths in configuration space with fixed end points

§1.2 Action functional

The answer given by the Hamilton's principle also known as the "Action Principle", is stated below. We first define action functional $\Phi(C)$ or (S_C). Given a path C, we known the coordinates as function of time and also the generalised velocities.at times between t_1 and t_2. Thus the Lagrangian $L(q, \dot{q}, t)$, for a given path, is expressible as a function of time t. This function of time when integrated over from t_1 to t_2 defines the action functional $\Phi(c)$ for the path C: [1]

$$\Phi(C) = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \tag{4}$$

Note that for a given system, the right hand is a number which depends on the path C, being different for different paths.

Now let C be a path which differs infinitesimally different from the path C. The path C' starts from q' at time t_1' and ends at q_2' at time t_2'. Let the values of coordinate be $q(t)$ at times between t_1' and t_2'. We will say that C' is infinitesimally different from the path C if the quantities defined by

$$\Delta t_1 = t_1' - t_; \qquad \Delta t_2 = t_2' - t_2 \tag{5}$$

$$\Delta q_1 = q_1' - q_1; \qquad \Delta q_2 = q_2' - q_2 \tag{6}$$

and

$$\delta q_1(t) = q_1'(t) - q(t), \qquad t_1 \leq t \leq t_2 \tag{7}$$

are infinitesimal quantities. For our present purpose it is unimportant weather we take (t_1, t_2) or (t_1', t_2') as the range of t in equation Eq.(4). The difference in velocities for the two paths is given by

$$\delta \dot{q}(1) = \frac{d}{dt} q'(t) - \frac{d}{dt} q(t) \tag{8}$$

$$= \frac{d}{dt}(\delta q(t)) \tag{9}$$

To formulate Hamilton's principle we compute variation of action functional

[1] A functional is a number assigned to function taken from class of functions. Here the functions are coordinates $q(t)$ as function of time.

when path is varied from C to C'

$$\begin{aligned}
&\Phi(C') - \Phi(C) && (10)\\
&= \int_{t'_1}^{t'_2} L(q'_1(t), \dot{q}(t), t)dt - \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)dt && (11)\\
&= \int_{t'_1}^{t_1} L(q'(t), \dot{q}'(t), t)dt + \int_{t_1}^{t_2} L(q', \dot{q}', t)dt + \int_{t_2}^{t'_2} L(q', \dot{q}, t)dt \\
&\qquad - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt && (12)\\
&\approx (t_1 - t'_1)L(q(t_1), \dot{q}(t_1), t) + \int_{t_1}^{t_2} \{L(q', \dot{q}', t) - L(q, \dot{q}, t)\}dt \\
&\qquad +(t_2 - t'_2)L(q_2, \dot{q}_2, t_2) && (13)\\
&\approx -\Delta t_1 L(q(t_1), \dot{q}(t_1), t_1) + \Delta t_2 L(q(t_2), \dot{q}(t_2), t_2 \\
&\qquad + \int_{t_1}^{t_2} \{L(q'(t), \dot{q}'(t), t) - L(q(t), \dot{q}(t), t)\}dt && (14)
\end{aligned}$$

Now we use the fact that the paths c' and c differ by infinitesimal amount (Eqn(9))

$$q'(t) = q(t) + \delta q(t) \tag{15}$$

substituting for $q'(t)$ in the last term of (14) We get

$$\begin{aligned}
&\int_{t_1}^{t_2} [L(q', \dot{q}', t) - L(q, \dot{q}, t)]\ dt && (16)\\
&= \int_{t_1}^{t_2} \left[L\Big(q + \delta q, \dot{q}(t) + \frac{d}{dt}\delta q(t)\Big) - L(q, \dot{q}, t)\right]\ dt && (17)\\
&\approx \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\Big)dt + \text{second order terms} && (18)\\
&= \int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k}\delta q_k - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k\right)\ dt + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big|_{t_1}^{t_2} && (19)\\
&\qquad \text{Integration by parts has been done in the second term} && (20)
\end{aligned}$$

substituting (19) in (14) we get

$$\Phi(C') - \Phi(C) \approx \int_{t_1}^{t_2} \Big(\sum_k \frac{\partial L}{\partial q_k} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\Big)\delta q_k + \Big[L\Delta t + \sum_k \frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big]_{t_1}^{t_2} \tag{21}$$

Hamilton's Principle: We first consider special class of variations of path which keep the end points fixed

$$\Delta t_1 = \Delta t_2 = 0 \tag{22}$$

$$\Delta q_k(t_1) = 0; \qquad \Delta q_k(t_2) = 0 \tag{23}$$

For such variations we get

$$\Delta\Phi(C) = \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\Big)\delta q_k(t)dt \tag{24}$$

It is now seen that the right the right hand side of(23) vanishes when ever Euler Lagrange equations are satisfied *i.e.* $q(t)$ is solution of EOM and the variation of action is zero implies that Euler Lagrange equations are satisfied. This is summarised into the following statement of action principle,

Action Principle: Given the configurations q_1, q_2 at times t_1 and t_2, the actual dynamical path C followed by a system is that for which the action is stationary *i.e.* C is that path about which infinitesimal variations do not produce any change in Φ

$$\delta\Phi = \Phi(C') - \phi(C) = 0.$$

Note that the variation in path should not change the end points of the path.

The Wees Action Principle This principle states that under general variations, end points may not be fixed, the dynamical path followed by the system is that the variations about it have only end point contribution

$$\Delta\Phi(C) = \Delta \int_{t_1}^{t_2} L(q,\dot{q},t)dt$$

$$= \sum_k \Big(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\Big) - H\Delta t\Big|_{t_1}^{t_2}$$

§2 Symmetries and conservation laws

The action principle is an elegant formulation of the laws of motion of a dynamical system. This formalism also provides an important connection between symmetries of Lagrangian and Conservation laws.One of the uses of the Conservation laws is in integration of equations of motion. In areas of physics, such as particle physics, where interactions were not known experimentally observed connection laws and selection rules and corresponding symmetry principles have been guiding principles towards building a theory.

§2.1 Symmetries:

Consider an infinitesimal transformation

$$\delta q_k \Rightarrow q'_k = q_k + \delta q_k \tag{25}$$

where δq_k is a specific variation under consideration

$$\delta q_k = \varepsilon \phi_k. \tag{26}$$

We say that the Lagrangian is invariant under a transformations (1),(2) if

$$L(q_k + \varepsilon\phi_k, \dot{q}_k + \varepsilon\dot{\phi}_k, t) - L(q_k, \dot{q}_k, t) = o(\varepsilon^2), \tag{27}$$

i.e. L.H.S has no terms proportional to ε.

An infinitesimal transformation type (1) applied to a path C in configuration space, gives rise to another path C' which is close to C. The corresponding variation in action can be computed from earlier result and is given by

$$\delta\Phi(c) = \varepsilon \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\Big)\phi_k(t) + \varepsilon \sum_k p_k \phi_k(t)\Big|_{t_1}^{t_2} \tag{28}$$

Eq.(27) implies $\Delta\Phi(c) = 0$. If C is the classical trajectory on which the EOM are obeyed, the integrand in the first term vanishes and we get

$$F(t_2) = F(t_1) \tag{29}$$

where

$$F(t) = \sum p_k \phi_k(t) = \sum \Big(\frac{\partial L}{\partial \dot{q}_k}\Big)\phi_k(t) \tag{30}$$

The equation (29) shows that $F(t)$ is independent of time, when ever EOM are obeyed, *i.e.* $F(t)$ is a constant of motion.

§2.2 Symmetry under a continuous transformation

Many times the Lagrangian has a geometric transformation which is easy to guess.Such a, finite, geometric transformation can be built up from infinitesimal transformations and the symmetry implies a Conservation Law We shall say that Lagrangian is "quasi invariant" when, instead of (3), we have

$$L(q_k + \varepsilon\phi_k, \dot{q}_k + \varepsilon\dot{\phi}_k, t) - L(q_k, \dot{q}_k, t) = \varepsilon\frac{d\Omega}{dt} \tag{31}$$

where Ω is a function of coordinates.In this case the L.H.S of (4) becomes $\sum_k \int_{t_1}^{t_2} \frac{d\Omega}{dt}$ and we get

$$\varepsilon\Omega(t)\Big|_{t_t}^{t_2} = \varepsilon \int_{t_1}^{t_2} \sum_k \Big(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}\Big)\phi_k dt + \varepsilon \sum p_k \phi_k(t)\Big|_{t_1}^{t_2} \tag{32}$$

or

$$\Big(\sum_k p_k\phi_k(t) - \Omega(t)\Big)\Big|_{t_1}^{t_2} = 0 \tag{33}$$

and the quantity

$$G(t) = \sum p_k\phi_k - \Omega \tag{34}$$

is a constant of motion. Several generalisations of transformations are possible we refer reader to the book by Sudarshan and Mukunda.

Noether's Theorem The result that has been obtained here is summarized as follows: associated with every continuous symmetry transformation of action, there exists a conserved quantity.

§3 Conservation of energy

We have seen that invariance of Lagrangian under translations,rotations leads to conservation laws for momentum and angular momentum respectively.What about conservation of energy? Is it also related to a symmetry of the Lagrangian? The answer is Yes: If the Lagrangian does not depend on time explicitly,there is a conservation law which is reduces to energy conservation for system with many particles. The conserved quantity will be called as Hamiltonian which reduces energy ($= KE + PE$) for a mechanical system,for large class of systems. For other system, qualifies to be identified with energy. If Lagrangian does not contain t explicitly, we have $\frac{\partial L}{\partial t} = 0$ and hence

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}(\ddot{q}_k) \tag{35}$$

using Euler Lagrange EOM we get

$$\frac{dL}{dt} = \sum_k \Big[\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k})\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k\Big] = \frac{d}{dt}\sum_{k=1} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k \tag{36}$$

or

$$\frac{d}{dt}\Big(\sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L\Big) = 0. \tag{37}$$

Hence H defined by

$$H \stackrel{\text{def}}{\equiv} \sum_{k=1} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \tag{38}$$

is a constant of motion. We can also write

$$H = \sum_{k=1}^{N} p_k\dot{q}_k - L \tag{39}$$

where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is called the **canonical momentum conjugate** to the coordinate $\dot{q}_k$and H will be called **Hamiltonian** of the system

In an alternate form of dynamics, the canonical momenta take over the role played by velocities and Hamiltonian becomes central quantity which governs the dynamics. The EOM can be written in an alternate form called the Hamiltonian EOM.

Example: Let us consider a single particle moving in force field described by potential energy $V(\vec{r})$.Then

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r}); \vec{r} = (x, y, z) \tag{40}$$

the canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}; \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}; \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}. \tag{41}$$

and the Hamiltonian is given by

$$\begin{aligned} H &= \sum p_k \dot{q}_k - L & (42)\\ &= (p_x m\dot{x} + p_y m\dot{y} + p_z m\dot{z}) - L & (43)\\ &= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(\vec{r})\right] & (44)\\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(\vec{r}) & (45)\\ &= \frac{1}{2}m\dot{\vec{r}}^{\,2} + V(\vec{r}) & (46) \end{aligned}$$

Thus the Canonical momenta, in this example,coincide with components of momentum $m\dot{\vec{r}}$ and Hamiltonian is equal to the energy

However, it must be remarked that the canonical momenta are not always equal to ‘ordinary’ momenta and Hamiltonian need not be a sum of kinetic and potential energy. When the system is described by a velocity dependent generalized potential Example will appear in problem sets.

§4 Noether’s theorem examples

For a system of N particles interacting via potential $V(|\vec{x}_\alpha - \vec{x_\beta}|)$ find symmetries and conservation laws

$$L(x, \dot{x}) = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{x}}_\alpha^2 - \sum_{\alpha<\beta} V(|\vec{x_\alpha} - \vec{x_\beta}|) \tag{47}$$

§4.1 Translation symmetry

Consider the translation transformations

$$\vec{x}_\alpha \to \vec{x}'_\alpha = \vec{x}_\alpha - \vec{a}. \tag{48}$$

Then we have

$$\dot{\vec{x}}'_\alpha = \dot{\vec{x}}_\alpha, \qquad |\vec{x}'_\alpha - \vec{x}'_\beta| = |\vec{x}_\alpha - \vec{x}_\beta|. \tag{49}$$

Translations are a symmetry transformations of the Lagrangian (47). To see this consider

$$L(\vec{x}', \dot{\vec{x}}') = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{x}}'^2_\alpha - \sum_{\alpha,\beta} V(|\vec{x}'_\alpha - \vec{x}'_\beta|) \tag{50}$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{x}}'^2_\alpha - \sum_{\alpha,\beta} V(|\vec{x}_\alpha - \vec{x}'_\beta|) \tag{51}$$

$$= L(\vec{x}, \dot{\vec{x}}) \tag{52}$$

translations is

$$\sum \frac{\partial L}{\partial \dot{q}_\alpha}\delta q_\alpha = \sum m_\alpha \dot{\vec{x}_\alpha}.\vec{a} = \vec{a}.\sum m_\alpha \dot{\vec{x}_\alpha} \tag{53}$$

$$\frac{d}{dt}\vec{a}.\vec{p} = 0 \Rightarrow \vec{a}.\frac{d}{dt}\vec{p} = 0 \tag{54}$$

since $\vec{a}$ is arbitrary we get

$$Therefore, Lagrangianisinvariant.Theconservedquantitycorrespondingto\frac{d}{dt}\vec{p} = 0. \tag{55}$$

where $\vec{p} = \sum_k m_\alpha \dot{\vec{x}}_\alpha$ is the total momentum.
Thus we see that invariance under translations gives rise to conservation of total momentum.

§4.2 Rotations

Let a set of axes K' be obtained from a set K by applying rotation about X_3 axis by an angle θ' then

A Figure is to be Drawn

Fig. **2** Rotation

$$x'_{\alpha 1} = x_{\alpha 1}\cos\theta + x_{\alpha 2}\sin\theta \tag{56}$$

$$x'_{\alpha 2} = -x_{\alpha 1}\sin\theta + x_{\alpha 2}\cos\theta \tag{57}$$

$$x'_{\alpha 3} = x_{\alpha 3} \tag{58}$$

Note that

$$
\begin{aligned}
& \dot{x}'^2_{\alpha 1} + \dot{x}'^2_{\alpha 2} + x'^2_{\alpha 3} && (59)\\
&= (\dot{x}_{\alpha 1}\cos\theta + \dot{x}_{\alpha 2}\sin\theta)^2 + (-\dot{x}_{\alpha 1}\sin\theta + \dot{x}_{\alpha 2}\cos\theta)^2 + x^2_{\alpha 3} && (60)\\
&= \dot{x}^2_{\alpha 1} + \dot{x}^2_{\alpha 2} + \dot{x}^2_{\alpha 3} && (61)
\end{aligned}
$$

Similarly,

$$|\vec{x}_\alpha{}' - \vec{x_\beta}{}'|^2 = |\vec{x}_\alpha - \vec{x_\beta}|^2 \qquad \dot{\vec{x}}'^2_\alpha = \vec{x}^2_\alpha \tag{62}$$

This implies

$$L(x'_\alpha, \dot{x}'_\alpha) = \frac{1}{2}\sum m_\alpha \dot{\vec{x}}^2_\alpha + V(|x_\alpha - x_\beta|) = L(x_\alpha, \dot{x}_\alpha) \tag{63}$$

Therefore, the Lagrangian is invariant under rotations. Let us now compute the variations for small θ:

$$
\begin{aligned}
\delta x'_{\alpha 1} &= x_{\alpha 1}\cos\theta + x_{\alpha 2}\sin\theta - x_{\alpha 1} && (64)\\
&\cong \theta x_{\alpha 2} + O(\theta^2) && (65)\\
\delta x'_{\alpha 2} &= -x_{\alpha 1}\sin\theta + x_{\alpha 2}\cos\theta - x_{\alpha 2} \simeq -\theta x_{\alpha 1}; && (66)\\
\delta x_{\alpha 3} &= 0 && (67)\\
& && (68)
\end{aligned}
$$

Therefore conserved quantity is

$$
\begin{aligned}
G &= \sum_{\alpha,k} \frac{\partial L}{\partial \dot{x}_{\alpha,k}} \delta x_{\alpha k} && (69)\\
&= \sum_\alpha \Big(\frac{\partial L}{\partial \dot{x}_{\alpha 1}}\delta x_{\alpha 1} + \frac{\partial L}{\partial \dot{x}_{\alpha 2}}\delta x_{\alpha 2} + \frac{\partial L}{\partial \dot{x}_{\alpha 3}}\delta x_{\alpha 3} \Big) && (70)\\
&= \theta \sum_\alpha \Big(m_\alpha \dot{x}_{\alpha 1} x_{\alpha 2} - m_\alpha \dot{x}_{\alpha 2} x_{\alpha 1} \Big) && (71)\\
&= \theta \sum_\alpha m_\alpha \Big(\dot{x}_{\alpha 1} x_{\alpha 2} - \dot{x}_{\alpha 2} x_{\alpha 1} \Big) && (72)\\
&= \theta \sum_\alpha \Big(\dot{p}_{\alpha 1} x_{\alpha 2} - \dot{p}_{\alpha 2} x_{\alpha 1} \Big) && (73)\\
&= \theta \sum_\alpha \Big(\vec{x}_\alpha \times \vec{p}_\alpha \Big)_3 && (74)
\end{aligned}
$$

$$\frac{dG}{dt} = 0 \Rightarrow \frac{d}{dt}\sum_\alpha \Big(\vec{x}_\alpha \times \vec{p}_\alpha \Big)_3 = 0 \tag{75}$$

Therefore, the third component of angular momentum $L_3 = \sum_\alpha \Big(\vec{x}_\alpha \times \vec{p}_\alpha \Big)_3$ is a constant of motion.
Similarly, invariance under rotations about other axes leads to conservation of other components of momentum.

§4.3 Galilean Transformation

The Galilean transformation from a frame to another frame moving with velocity $\vec{v}$ is given by a frame

$$\vec{x}\,' = \vec{x} - \vec{v}t$$

where $\vec{v}$ is independent of time. Therefore,

$$\dot{\vec{x}}\,' = \dot{\vec{x}} - \vec{v}$$

and

$$\delta\vec{x}\,' = -vt; \qquad \delta\dot{\vec{x}}\,' = -v.$$

Since $V(|\vec{x}\,'_\alpha - \vec{x}\,'_\beta|) = V(|\vec{x}_\alpha - \vec{x}_\beta|)$, the change in Lagrangian is given by

$$\begin{aligned}
\delta L &= L(\vec{x}\,', \dot{\vec{x}}\,') - L(\vec{x}, \dot{\vec{x}}) && (76)\\
&= \frac{1}{2}\sum m_\alpha \dot{\vec{x}}\,'^2_\alpha - \frac{1}{2}\sum m_\alpha \dot{\vec{x}}^2_\alpha && (77)\\
&= \frac{1}{2}m(\dot{\vec{x}}\,'^2 - \vec{v})^2 - \frac{1}{2}m\dot{\vec{x}}^2_\alpha && (78)\\
&= -m_\alpha \dot{\vec{x}_\alpha}\vec{v} + O(v^2). && (79)
\end{aligned}$$

Thus

$$L(\vec{x}\,', \dot{\vec{x}}\,') = L(\vec{x}_\alpha, \dot{\vec{x}_\alpha}) - \underbrace{\frac{d}{dt}\sum m_\alpha \vec{x}_\alpha.\vec{v}}_{\frac{d}{dt}\Omega} \qquad (80)$$

and we have identified Ω as indicated above. Therefore, conserved quantity is

$$\begin{aligned}
G &= \sum \frac{\partial L}{\partial \dot{x}_{\alpha k}}\delta x_{\alpha k} - \Omega && (81)\\
&= -\sum_\alpha \dot{x}_{\alpha k} v_k t + \sum m_\alpha \vec{x}_\alpha.\vec{v} && (82)\\
&= -\vec{v}\Big(\sum m_\alpha \dot{x}_{\alpha k} t - \sum m_\alpha \vec{x}_\alpha\Big) && (83)
\end{aligned}$$

Therefore, the quantity

$$\vec{v}\Big(\sum m_\alpha \dot{\vec{x}}_{\alpha k} t - \sum m_\alpha \vec{x}_\alpha\Big)$$

is independent of time. To understand it a little, let us introduce C.M coordinate $\vec{X}$ then we get

$$\vec{v}\Big(\dot{\vec{X}}t - \vec{X}\Big) = \text{constant}$$

or

$$\vec{X} = \dot{\vec{X}}t + \text{constant}$$

Therefore $\vec{X}$ is a linear function of time. Thus the conservation law associated with invariance of Galilean transformations is that the centre of mass of the system moves with a constant velocity $\dot{\vec{X}}$.