§3.3 Dirac Bra and Ket Notation

Dirac Bra-Ket Notation

In this lecture I will explain the Dirac Bra Ket notation for vector spaces with inner product. This notation is extremely useful for quantum mechanics. When an o.n. basis is selected in the vector space Dirac notation is very convenient and several formulas such concerning representations and change of basis become simple and easy to remember.

A vector in a vector space will be called "ket vector". In Dirac notation a vector f is written as $|f\rangle$. A linear functional, ψ, is an element of the dual vector space and is written as $\langle\psi|$ and will be called a "bra vector"

In inner prodcut spaces there is a correspondence between the vector space and its dual space of linear functionals. To every given vector $|h\rangle$ we associate a linear functional, a bra vector to be denoted by $\langle h|$. The notation $\langle h|f\rangle$ will denotes the action of the linear functional $\langle h|$ on a vector $|f\rangle$. There is no need to introduce a separate notation for the scalar product of vector $|h\rangle$ with $|f\rangle$; it will be identified with $\langle h|f\rangle$.

Let $\mathcal{E} = |e_1\rangle, |e_2\rangle, ..., |e_N\rangle$ be an o.n. basis. If a vector $|\psi\rangle$ is written as linear combination of the basis elements in $\mathcal{E}$,

$$|\psi\rangle = \sum \alpha_k |e_k\rangle$$

the coefficients will be given by the scalar products $\alpha_k = \langle e_k|\psi\rangle$. Substituting the value of α we can write the expansion of $|\psi\rangle$ as

$$|\psi\rangle = \sum_k |e_k\rangle\langle e_k|\psi\rangle$$

The vector $|e_k\rangle\langle e_k|\psi\rangle$ appearing inside the sum can be thought of as a linear operator $T_k, (\equiv |e_k >< e_k|)$, which on the vector $|\psi\rangle$ gives $|e_k\rangle\langle e_k|\psi\rangle$.

$$T_k|\psi\rangle = |e_k\rangle\langle e_k|\psi\rangle$$

The relation can be viewed as a statement that the relation $|\psi\rangle = \sum T_k|\psi\rangle$ holds for every vector $|\psi\rangle$. Thus $\sum T_k$ must be equal to identity operator. Hence we get

$$\sum_k |e_k\rangle\langle e_k| = I$$

This relation is referred to as *completeness relation.*

Change Of O.N. Basis

Let x be a vector in a vector space. Let $\mathcal{E}$ and $\mathcal{U}$ be two o.n. bases. Let $\underline{\mathsf{x}}$ and $\underline{\underline{\mathsf{x}}}$ denote the components of the vector x w.r.t. the bases $\mathcal{E}$ and $\mathcal{U}$ respectively. Similarly let $\underline{\mathsf{T}}$ denote the matrix representing an operator T w.r.t. the first basis $\mathcal{E}$. Let $uusfT$ be the matrix w.r.t. the second basis $\mathcal{U}$. Let us take the first o.n. basis as $\mathcal{E} = e_1, e_2, \ldots, e_N$ then we have the following expressions.

$$\underline{\mathsf{x}}_k = \langle e_k|x\rangle, \qquad [\underline{\mathsf{T}}]_{jk} =< e_j|T|e_k > \tag{70}$$

If we take the second o.n. basis as $\mathcal{U} = \{u_1, u_2, ..., u_N\}$ and we have the following expressions.

$$\underline{\underline{\mathsf{x}}}_i =< u_i|x >, \qquad [\underline{\underline{\mathsf{T}}}]_{jk} =< u_j|T|u_k > \tag{71}$$

We want to find relations between

(i) components of $\underline{x}$ and $\underline{\underline{x}}$,

(ii) elements of the matrices $\underline{T}$ and $\underline{\underline{T}}$. The change of basis can be achieved by using the completeness relation. For example

$$\underline{x}_i = \langle e_i|x\rangle = \langle e_i|I|x\rangle \tag{72}$$
$$= \langle e_i|\Big(\sum_k |u_k\rangle\bar{u_k}\Big)|x\rangle \tag{73}$$
$$= \sum_k \langle e_i|u_k\rangle\langle u_k|x\rangle \tag{74}$$
$$= \sum_k \langle e_i|u_k\rangle\underline{\underline{x}}_k \tag{75}$$

This gives the required relation between the components of the vector x w.r.t. the two basis sets $\mathcal{E}$ and $\mathcal{U}$. Similarly,

$$[\underline{T}]_{jk} = \langle e_j|T|e_k\rangle \tag{76}$$
$$= < e_j|\Big(\sum_m |u_m\rangle\langle u_m|\Big)\, T\, \Big(\sum_n |u_n\rangle\langle u_n|\Big)|e_k\rangle > \tag{77}$$
$$= \sum_m\sum_n \langle e_j|u_m\rangle\langle u_m|T|u_n\rangle\langle u_n|e_k\rangle \tag{78}$$
$$= \sum_m\sum_n \langle e_j|u_m\rangle[\underline{\underline{T}}]_{mn}\langle u_n|e_k\rangle \tag{79}$$
$$\tag{80}$$

This gives the change of basis formula for the matrices representing the operators.

All These Results Are Valid For Finite Dimensional Vector Spaces Only. Their Use In Case Of Infinite Dimensional Vector Spaces Requires A Separate Detailed Discussion.

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