## §2.3 Hermitian Operators

**Definition 10** *An operator $A$ is* **hermitian** *if $A^\dagger = A$.*

**Definition 11** *A linear operator $U$ is called* **unitary** *if $U^\dagger = U^{-1}$. In case of a finite dimensional vector space, it is equivalent to demanding*

$$UU^\dagger = I(or U^\dagger U = I).$$

**Theorem 8 (When is an operator hermitian ?)** *Each of the following two statements give condition for hermiticity of an operator.*

1. An operator $T$ is hermitian if and only if $(Tg, f) = (g, Tf)$ holds for all $f, g \in \mathcal{V}$.
2. In a finite dimensional vector space an operator $T$ is self adjoint if and only if $(f, Tf)$ is real $\forall f \in \mathcal{V}$.

**Proof of (1)** : Let T be a hermitian operator. Using the definition of adjoint we have

$$(g, Tf) = (T^\dagger g, f)$$

or

$$(g, Tf) = (Tg, f) \qquad (\because T = T^\dagger)$$

Let $(g, Tf) = (Tg, f)$ for all f and g in the vector space. Then we get

$$
\begin{aligned}
(g, Tf) &= (Tg, f) \qquad \text{(given )} && (41)\\
(g, Tf) &= (g, T^\dagger f) \quad \text{(Use def of } T^\dagger) && (42)\\
(g, (T - T^\dagger)f) &= 0 && (43)
\end{aligned}
$$

holds $\forall g$ and $f$. Select $g = (T - T^\dagger)f$. This gives $\|(T - T^\dagger)f\| = 0$. Therefore,

$$(T - T^\dagger)f = 0 \qquad \forall f \in \mathcal{V}. \tag{44}$$

Hence $T = T^\dagger$

**Proof of (2)** : Let $(f, Tf)$ be real. Then

$$
\begin{aligned}
(f, Tf) &= (f, Tf)^* \qquad \text{given} && (45)\\
&= (Tf, f) \qquad \text{( property of inner product )} && (46)\\
&= (f, T^\dagger f) \qquad \text{( def of adjoint )} && (47)
\end{aligned}
$$

Thus $(f, Tf) = (f, T^\dagger f)$ holds $\forall f \in \mathcal{V}$. This implies $(f, (T - T^\dagger)f) = 0$, hence $T = T^\dagger$. Therefore, $T$ is hermitian.

**Theorem 9** *If $X$ is any operator we may write, $X = A + iB$, where $A$ and $B$ are hermitian operators.*

The proof is easy. We write $A = (X + X^\dagger)/2; \qquad B = (X - X^\dagger)/2i$ It is straight forward to verify that $A$ and $B$ are hermitian and that $X = A + B$.

**Theorem 10 (Eigen-values and Eigen-vectors of Hermitian Operators)** *Two important properties of hermitian operators are given below.*

1. The eigen-values of a hermitian operators are real.
2. The eigen-vectors of a hermitian operator corresponding to two distinct eigen-values are orthogonal.

**Proof of 1** : Let $\lambda$ be an eigen-value and $f$ be eigen-vector of $T$ with $Tf = \lambda f$. Taking $g = f$ in $(f, Tg) = (Tf, g)$ we get

$$(Tf, f) = (f, Tf) \tag{48}$$

$$(\lambda f, f) = (f, \lambda f) \tag{49}$$

$$\therefore \quad (\lambda^* - \lambda)(f, f) = 0 \tag{50}$$

As $f \neq 0, (f, f) \neq 0$ and hence we must have $\lambda^* - \lambda = 0$ Therefore the eigen-values of a hermitian operator are real.

**Proof of 2** :To prove that two eigen-vectors corresponding to a different eigen-values are orthogonal. Let $Tf = \lambda f$ and $Tg = \mu g$. and $T$ be a hermitian operator $T\dagger = T$ and $\lambda \neq \mu$. Then proceeding as in proof of (1)

$$(f, Tg) = (Tf, g) \qquad (\text{Since } T^\dagger = T)$$

We have

$$(f, \mu g) = (\lambda f, g)$$

or

$$\mu(f, g) = \lambda^*(f, g) = \lambda(f, g)$$

because the eigenvalues $\lambda, \mu$ are real. For $\lambda \neq \mu$ the above equation implies that $(f, g) = 0$. Hence $f$ and $g$ are orthogonal.

**Definition 12** *An operator is called* **projection operator** *if* $P^2 = P$.

**Definition 13** *An operator is called* **positive** *if* $(f, Af) \geq \forall f \in \mathcal{V}$.