§1.2 Orthogonality and Gram Schmidt Procedure

Definition 3 *We say that two vectors f and g are* **orthogonal** *if* $(f,g)=0$

LEMMA : If $g \neq 0$ then the vector

$$x = f - \frac{(g,f)}{(g,g)} g$$

is orthogonal to g.

Proof :

Consider

$$(g,x) = (g, f - \frac{(g,f)}{(g,g)} g) = (g,f) - \frac{(g,f)}{(g,g)}(g,g) \quad (10)$$

$$= (g,f) - (g,f) = 0 \quad (11)$$

Therefore, g is orthogonal to $x = f - \frac{(g,f)}{(g,g)} g$.

Definition 4 *Two vectors f and g are* **orthogonal** *if* $(f,g)=0$.

Definition 5 *A set of vectors $\mathcal{X}$ is an* **orthogonal set** *if $\forall$ pair $x, y \in \mathcal{X}$, we have $(x,y)=0$.*

Definition 6 *A set of vectors $\mathcal{X}$ is called* **orthonormal set** *if*

(*a*) *for every pair $x, y \in \mathcal{X}$ we have $(x,y)=0$ and*

(*b*) *for every $x \in \mathcal{X}$ we have $\|x\| = 1$.*

Definition 7 *A set $\{x_1, x_2, ..., x_r\}$ is an* **orthonormal set** *iff $(x_i, x_j) = \delta_{ij}$.*

Definition 8 *An orthonormal set is called a* **complete orthonormal set** *if it is not contained in any larger orthonormal set.*

Theorem 1 *An orthogonal set $\mathcal{X} = \{x_1, x_2, ..x_r\}$ of non-zero vectors is linearly independent.*

Proof : Consider

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_r x_r = 0 \quad (12)$$

Taking scalar product with x_1 gives zero for all terms except the first one. Thus

$$\alpha_1 (x_1, x_1) = 0 \Rightarrow \alpha_1 = 0 \quad (13)$$

$$(\because x_1 \neq 0 \Rightarrow (x_1, x_1) \neq 0). \quad (14)$$

Remark : Earlier we have seen that the vector $h = f - \lambda g$ is orthogonal to the vector g if λ is taken to be $(g, f)/(g, g)$. The following theorem generalizes this result to orthogonal sets.

Theorem 2 *If $\mathcal{U} = u_1, u_2, ..., u_n$ is any finite orthogonal set containing nonzero vectors of an inner product space and if $\lambda_k = (u_k, x)/(u_k, u_k)$, then the vector h defined by*

$$h = f - \lambda_1 u_1 - \lambda_2 u_2 - ... - \lambda_k u_k$$

is orthogonal to every element u_k in the set $\mathcal{U}$

The result follows easily by taking the scalar products (h, u_k) for different k.

Grahm Schmidt Orthogonalization Procedure

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_r\}$ be a linearly independent set. Then one can construct a set of vectors $\mathcal{E} = \{e_1, e_2, ...e_r\}$ such that the vectors e_k are linear combinations of the vectors in $\mathcal{X}$ and the set $\mathcal{E}$ is an orthonormal set.

Proof: Define

$$\begin{aligned} u_1 =& x_1, & e_1 = u_1/\|u_1\| \\ u_2 =& x_2 - (e_1, x_2)e_2, & e_2 = u_2/\|u_2\| \\ u_3 =& x_3 - (e_1, x_3)e_3 - (e_2, x_3)e_2, & e_3 = u_3/\|u_3\| \\ u_r =& x_r - \sum_{k=1}^{r-1}(e_k, x_r)e_k, & e_r = u_r/\|u_r\| \end{aligned}$$

It is easily verified that $\{e_1, e_2, ...\}$ is an o.n. set.

Bessel's Inequality

If $\mathcal{U} = u_1, u_2, ..., u_r$ is any finite orthonormal set in an inner product space then for all $x \in \mathcal{V}$ we have

$$\sum_k |(u_k, x)|^2 \leq \|x\|^2 \qquad \text{(Bessel Inequality)} \tag{15}$$

Proof : For every vector y , we have $(y, y) \geq 0$. Therefore, taking y to be

$$y = x - \sum_k \lambda_k u_k \qquad \text{with } u_k = (u_k, x).$$

we get

$$(y, y) = (x - \sum_k \lambda_k u_k, x - \sum_j \lambda_j u_j) \tag{16}$$

$$= (x, x) - \sum_k \lambda_k^*(u_k, x) - \sum_j \lambda_j(x, u_j) + \sum_j \sum_k \lambda_k^* \lambda_j (u_j, u_k) \tag{17}$$

$$= (x, x) - \sum_k \lambda_k^*(u_k, x) - \sum_j \lambda_j(x, u_j) + \sum_k \lambda_k^* \lambda_k \tag{18}$$

One of two the summations in the last term has been done using $(u_j, u_k) = \delta_{jk}$. Substituting $\lambda_j = (u_j, x)$ we get

$$\begin{aligned} (y, y) &= (x, x) - \sum (x, u_k)(u_k, x) - \sum (u_j, x)(x, u_j) + \sum (x, u_j)(u_j, x) && (19) \\ &= (x, x) - \sum_k (x, u_k)(u_k, x) && (20) \\ &= (x, x) - \sum_k |(u_k, x)|^2 && (21) \end{aligned}$$

Using $(y, y) \geq 0$ we get the desired Bessel's inequality.

$$\sum_k \; |(u_k, x)|^2 \leq \|x\|^2 \tag{22}$$