§3.1 Basis and matrix representation

We shall now discuss a way of representing vectors in a vector space, having dimension N, by N component column vectors and linear operators by $N \times N$ matrices .

Let $\mathcal{X} = x_1, x_2, ..., x_N$ be a basis. Every vector $x \in \mathcal{V}$ can be expanded in terms of the basis vectors $x_1, x_2, ..., x_N$. Thus

$$x = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_N x_N. \tag{16}$$

The scalars $\xi_1, \xi_2, ..., \xi_N$ will be called the components of the vector x with respect to the basis $\mathcal{X}$. Knowing the vector x the scalars $\xi_1, \xi_2, ..., \xi_N$ are uniquely fixed and conversely if the scalars $\xi_1, \xi_2, ..., \xi_N$ are given, the vectors in the basis $\mathcal{X}$ can be used to get the vector. We shall assemble the components $\xi_1, \xi_2, ..., \xi_N$ in form of an N- component column denoted by x.

$$x \mapsto \mathsf{x} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ ... \\ \xi_N \end{pmatrix} \tag{17}$$

Let $x \in \mathcal{V}$ be a vector and T be a linear operator. The answers for the column representing a vector and for matrix representing an operator depends on chioce of basis, and will change when a new basis is selected.

To find the matrix representing a linear operator T, we note that the knowledge of the action of a linear operator on a set of basis vectors is sufficient to know the action of an operator on any vector. Thus we consider the basis $\mathcal{X} = x_1, x_2, ..., x_N$ and apply the operator T on every element to obtain the set $T\mathcal{X} = \{Tx_1, Tx_2, Tx_3, ...Tx_N\}$. Next we expand the vectors in the set Tx_k so obtained in terms of the basis vectors.

$$Tx_k = \sum_j t_{jk} x_j, \qquad k = 1, 2, ..., \tag{18}$$

The m^{th} row and n^{th} column of the matrix T is given by t_{mn}. We write the above N equations for $k = 1, 2, ..., N$ as

$$\begin{align} Tx_1 &= t_{11}x_1 + t_{21}x_2 + t_{31}x_3 + ... + t_{N1}x_N \tag{19}\\ Tx_2 &= t_{12}x_1 + t_{22}x_2 + t_{32}x_3 + ... + t_{N2}x_N \tag{20}\\ Tx_3 &= t_{13}x_1 + t_{23}x_2 + t_{33}x_3 + ... + t_{N3}x_N \tag{21}\\ Tx_N &= t_{1N}x_1 + t_{2N}x_2 + t_{3N}x_3 + ... + t_{NN}x_N \tag{22}\\ & \tag{23} \end{align}$$

The rule for constructing the matrix T for the operator T is to collect the coefficients appearing in the above equations as a matrix and take its transpose. Thus we have

$$T \to \mathsf{T} = \text{Transpose of} \begin{bmatrix} t_{11} & t_{21} & t_{31} & \cdots & t_{N1} \\ t_{12} & t_{22} & t_{32} & \cdots & t_{N2} \\ t_{13} & t_{23} & t_{33} & \cdots & t_{N3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{1N} & t_{2N} & t_{3N} & \cdots & t_{NN} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1N} \\ t_{21} & t_{22} & t_{23} & \cdots & t_{2N} \\ t_{31} & t_{32} & t_{33} & \cdots & t_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{N1} & t_{N2} & t_{N3} & \cdots & t_{NN} \end{bmatrix} \tag{24}$$

Thus $\mathsf{T}_{ij} = \mathsf{t}_{ij}$. With this every vector space of dimension N becomes isomorphic to $\mathbb{F}^N$. Every relation between vectors and operators is equivalent to a relation between $N-$ component columns and $N \times N$ matrices. For example

If $y = Tx$ we have $\mathsf{y}= \mathsf{T}\ \mathsf{x}$; Similarly, if $AB = C$ then $\mathsf{A} = \mathsf{BC}$ where A, B, C are operators and $x, y, ..$ are vectors in $\mathcal{V}$.

An Example

- Let $e_1 = (1, 1, 0), e_2 = (0, 1, 1), e_3 = (1, 0, 1)$ be a basis in $\mathbb{R}^3$. Find components of a vector $f = (x, y, z)$ and represent it w.r.t. the basis $\{e_1, e_2, e_3\}$
- Let an operator T be defined as $Te_1 = (1, 0, 0), Te_2 = (0, 1, 0)$ and $Te_3 = (1, 1, 1)$ Knowledge of action of an operator T on a basis is sufficient to find its action on any vector. Given e_1, e_2, e_3 as above, find the vector $g = Tf$. 3. Find the representatives of two vectors f, g and the matrix T w.r.t. the basis (e_1, e_2, e_3) and verify that $\mathsf{T}\ \mathsf{f} = \mathsf{g}$

SOLUTION :

(1) Let $f = (x, y, z)$ be written as a linear combination of the vectors e_1, e_2, e_3 :

$$\begin{aligned} f &= ae_1 + be_2 + ce_3 && (25)\\ (x, y, z) &= a(1, 1, 0) + b(0, 1, 1) + c(1, 0, 1) && (26)\\ &= (a + c, a + b, b + c) && (27)\\ \text{or} a + c = x; \qquad a + b &= y; \qquad b + c = z && (28)\\ &&& (29) \end{aligned}$$

This is easily solved to give

$$a = (x + y - z)/2, b = (y + z - x)/2, c = (z + x - y)/2.$$

$$(x, y, z) = \frac{(x + y - z)}{2} e_1 + \frac{(y + z - x)}{2} e_2 + \frac{(z + x - y)}{2} e_3$$

Thus $f \to \mathsf{f}$ where

$$f \mapsto \mathsf{f} \;=\; \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{30}$$

$$\mathsf{f} \;=\; \begin{pmatrix} \frac{(x+y-z)}{2} \\ \frac{(y+z-x)}{2} \\ \frac{(z+x-y)}{2} \end{pmatrix} \tag{31}$$

(2) To find how T acts on a general vector $h = ae_1 + be_2 + ce_3$, we compute $g = Th$. Using the

linearity property we get

$$\begin{aligned} g &= T[ae_1 + be_2 + ce_3] \tag{32}\\ &= aTe_1 + bTe_2 + cTe_3 \tag{33}\\ &= a(1,0,0) + b(0,1,0) + c(1,1,1) \tag{34}\\ &= (a+c, b+c, c) \tag{35}\\ \therefore Tf &= \left(x, z, \frac{(z+x-y)}{2}\right) \tag{36} \end{aligned}$$

For later use we remark that the vector g written as linear combination of $\{e_1, e_2, e_3\}$ becomes

$$g = \frac{(x+y+z)}{4} e_1 + \frac{(3z-x-y)}{4} e_2 + \frac{(3x-y-z)}{4} e_3.$$

(3) We construct the matrix for the operator w.r.t. the basis (e_1, e_2, e_3). For this purpose we must express Te_1, Te_2 and Te_3 as linear combinations of e_1, e_2, e_3.

$$\begin{aligned} Te_1 = (1,0,0) &= \frac{1}{2}e_1 - \frac{1}{2}e_2 + \frac{1}{2}e_3 \tag{37}\\ Te_2 = (0,1,0) &= \frac{1}{2}e_1 + \frac{1}{2}e_2 - \frac{1}{2}e_3 \tag{38}\\ Te_3 = (1,1,1) &= \frac{1}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \tag{39}\\ & \tag{40} \end{aligned}$$

Therefore, the matrix, T, respresenting the operator T w.r.t. the basis $\{e_1, e_2, e_3\}$ is given by

$$\begin{aligned} \mathsf{T} &= \text{Transpose of} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{41}\\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{42} \end{aligned}$$

A second way of computing the vector g is to apply T on f to get g as follows.

$$\mathsf{g} = \mathsf{T}\mathsf{f} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} (x+y-z)/2 \\ (y+z-x)/2 \\ (z+x-y)/2 \end{bmatrix} \tag{43}$$

or

$$g = \begin{bmatrix} (x+y-z)/4 \\ (3z-x-y)/4 \\ (3x-y-z)/4 \end{bmatrix} \tag{44}$$

This gives the components of the vector g w.r.t. the basis $\{e_1, e_2, e_3\}$. To get back the vector we recnostruct the vector g as

$$
\begin{aligned}
g &= \frac{(x+y+z)}{4}e_1 + \frac{(3z-x-y)}{4}e_2 + \frac{(3x-y-z)}{4}e_3 && (45)\\
&= \left(\frac{(x+y+z)}{4}, \frac{(x+y+z)}{4}, 0\right) && (46)\\
&\quad + \left(0, \frac{(3z-x-y)}{4}, \frac{(3z-x-y)}{4}\right) && (47)\\
&\quad + \left(\frac{(3x-y-z)}{4}, 0, \frac{(3x-y-z)}{4}\right) && (48)\\
&= \left(x, z, \frac{(z+x-y)}{2}\right) && (49)
\end{aligned}
$$

which agrees with the result obtained above.