§2.2 Linear operators

Definition 15 *An* **operator***, T, on a vector space $\mathcal{V}$ is a mapping*

$$T : \mathcal{V} \to \mathcal{V}$$

from the vector space $\mathcal{V}$ into itself. In other words, to an arbitrary vector f from the vector space, an operator, T, assigns a unique vector, Tf, in the vector space $\mathcal{V}$.

$$T : f \mapsto Tf \in \mathcal{V}$$

Definition 16 *An operator, T on a vector space is a* **linear operator** *if it satisfies the property*

$$T(\alpha + \beta g) = \alpha\ T + \beta\ Tg$$

$\forall \alpha, \beta \in \mathcal{F}$ and $\forall\ g \in \mathcal{V}$. Equivalently an operator T is linear if

$$T(f + g) = Tf + Tg \text{ and } T(\alpha f) = \alpha Tf$$

It is, therefore, seen that for an operator T to be linear it is necessary that $Tf = 0$ if $f = 0$.

Definition 17 *Given two linear operators A and B we can define their* **sum** *, $A+B$, by means of the following rule for its action on an arbitrary vector.*

$$(A + B)f = Af + Bf$$

The sum of two linear operators is again a linear operator.

Definition 18 Multiplication of a linear operator *T by a scalar α is a linear operator defined by*

$$(\alpha T)f = \alpha(Tf)$$

Theorem 3 *With addition of linear operators and scalar multiplication defined as above, the set of all linear operators on a vector space $\mathcal{V}$ is again a vector space. If the dimension of the vector space $\mathcal{V}$ is n, the dimension of the vector space of all the operators on$\mathcal{V}$ is n^2.*

Definition 19 *With sum, product and multiplication by a scalar defined for operators, the following expression defines* **polynomial** *in a linear operator A.*

$$p(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + + \alpha_n A^n$$

The operator $p(A)$ is again a linear operator.

Definition 32 *Let λ be an eigen-value of an operator T. Let $\nu(\lambda)$ denote the number of linearly independent eigen-vectors $Tx = \lambda x$. If $\nu(\lambda) = 1$ we say that the eigen-value λ is* **non-degenerate**. *When $\nu(\lambda) > 1$, we say that the eigen-value λ is* **degenerate** *and the* **degeneracy** *of the eigen-value λ is defined to be equal to the number of linearly independent eigen-vectors with eigen-value λ.*

Definition 26 *Let T be an operator which is both one to one and onto. We define* **inverse** *of T by giving its action on an arbitrary vector $u \in \mathcal{V}$.*

Because T is onto, we can find a vector u such that $Tu = v$. Since T is one to one it follows that u satisfying $Tu = v$ is uniquely determined once the vector v is specified. We define **inverse** of T, to be denoted by T^{-1}, by the equation

$$T^{-1}v = u$$

This definition coincides with the defintion of the inverse for a mapping. The inverse satisfies

$$(AB)^{-1} = B^{-1}A^{-1},$$

$$(\alpha A)^{-1} = (1/\alpha)A^{-1}, \alpha \neq 0.$$

Definition 27 *Let T be a linear operator on a vector space $\mathcal{V}$. The* **range** *$\mathcal{R}(T)$ is the set of vectors obtained by applying T on all vectors $f \in \mathcal{V}$.*

$$\mathcal{R}(T) = \{g | g = Tf, \in \mathcal{V}\}$$

Definition 28 *Also the* **null space** *of an operator, $\mathcal{N}(T)$, is the set of all those vectors x for which $Tx = 0$.*

$$\mathcal{N}(\ T) = \{x | x \in \mathcal{V} \text{ and } Tx = 0\}$$

Both $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are subspaces of the vector space $\mathcal{V}$. (Proof ?!)

Definition 29 *The dimension of $\mathcal{R}(T)$ for an operator is called the rank of the operator T. Obviously rank $(T) \leq \dim \mathcal{V}$.*

Clearly rank of an operator is the maximum number of linearly independent vectors that can be selected from Tf when varies over the entire vector space $\mathcal{V}$.

Eigen-values and Eigen-vectors

Definition 30 *A subspace $\mathcal{M} \subset \mathcal{V}$ is said to be an* **invariant subspace** *of an linear operator X if $\forall f \in \mathcal{M} Xf \in \mathcal{M}$.*

Definition 31 *Let T be linear operator. If f is a non-zero vector satisfying*

$$Tf = \lambda f$$

or some scalar λ , we say that f is an **eigen-vector** *of operator T and λ is the corresponding* **eigen-value.**

Note that $f = 0$ will always satisfy the equation $Tf = \lambda f$ for an arbitrary λ. Therefore, null vector is, by definition, excluded from being an eigen-vector.

It is possible that for a given λ there are more than one eigen-vectors satisfying the eigen-value equation $Tf = \lambda$. Therefore, we define

Product and Commutator

Definition 20 **Product of two operators**, *A and B, is defined as in the case of mappings.*

$$(AB)f = A(Bf)$$

When A and B are linear operators, the product AB is also a linear operator.

Definition 21 *The* **commutator** *of two operators is defined to be*

$$[\,A, B\,] = AB - BA$$

Definition 22 *The* **anticommutator** *is defined by*

$$[\,A, B\,]_+ = AB + BA$$

PROPERTIES OF COMMUTATOR

The commutator satisfies the following properties.

$$[\,A, B\,] = -[\,B, A\,] \tag{9}$$

$$[\,\alpha_1 A_1 + \alpha_2 A_2, B\,] = \alpha_1 [\,A_1, B\,] + \alpha_2 [\,A_2, B\,] \tag{10}$$

$$[\,A, \beta_1 B_1 + \beta_2 B_2\,] = \beta_1 [\,A, B_1\,] + \beta_2 [\,A, B_2\,] \tag{11}$$

$$[\,A, BC\,] = B[\,A, C\,] + [\,A, B\,]C \tag{12}$$

$$[\,AB, C\,] = A[\,B, C\,] + [\,A, C\,]B \tag{13}$$

$$[\,A, [\,B, C\,]] + [\,B, [\,C, A\,]] + [\,C, [\,A, B\,]] = 0 \tag{14}$$

The last relation is known as the Jacobi identity.

Inverse of an Operator

Definition 23 *Let T be an operator on a vector space. We say T is* **one to one** *if action of T on two* distinct vectors *gives* distinct answers.

$$x_1 \neq x_2 \implies Tx_1 \neq Tx_2$$

This is equivalent to the condition

$$Tx_1 = Tx_2 \Longrightarrow x_1 = x_2$$

Definition 24 *An operator T on a vector space is called* **onto** *if $\forall\ y \in \mathcal{V}$ we can find at least one $x \in \mathcal{V}$ such that $Tx = y$. This x may, in general, not be unique.*

Definition 25 *An operator is called* **invertible** *if it is both one to one and onto.*