§1.3 Linear combination, basis and dimension

Definition 6 (Linearly dependent set) *A set of vectors* $\mathcal{S} = \{f_1, f_2, ...f_n\}$ *is called linearly dependent set if* $\exists$ *a set of scalars* $\alpha_1, \alpha_2, \ldots$ *such that not all* α*'s are zero and*

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$$

Definition 7 (Linearly independent set) *A set of vectors* $\mathcal{S} = \{f_1, f_2, ...f_n\}$ *is called linearly independent set if it is not a linearly dependent set. This means that a set* X *is linearly independent if*

$$\alpha_1 f_2 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$$

implies $\alpha_1 = \alpha_2 = \cdots = 0$.

Definition 8 (Linear combination) *Let* $\{f_1, f_2, ..., f_m\}$ *be a finite set of vectors in vector space V. Let* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *be a set of scalars and* $f \in \mathcal{V}$ *be such that*

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$$

Then we say that f *is linear combination of the vectors* $f_1, f_2, \ldots, f_m$.

Properties Of Linear Combination

1. If $f \in \mathcal{V}$ is a linear combination of $\{f_1, f_2, \ldots\}$, then the scalars α_i in

$$f = \sum \alpha_i f_i$$

   are uniquely determined if and only if $\{f_1, f_2, \ldots\}$ is a independent set.

2. If $\{f_i\}$ is a linearly independent set, a necessary and sufficient condition that $f \in \mathcal{V}$ be a linear combination of $\{f_i\}$ is that the set $\{f, f_i\}$ be linearly dependent.

3. Every set of vectors containing a linearly dependent set is also linearly dependent.

Definition 9 (Finite dimensional space) *A vector space is called* **finite dimensional** *if* $\exists$ *an integer* N *such that every set containing more than* N *elements is a linearly dependent set.*

Definition 10 (Basis) *A set of vectors* $\mathcal{X}$ *is called a* **basis** *in a vector space* $\mathcal{V}$ *if the following two properties are satisfied.*

- the set $\mathcal{X}$ is a linearly independent set, and
- every vector $f \in \mathcal{V}$ is a linear combination of vectors in $\mathcal{X}$, i.e.,

$$f = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$$

  where $x_k \in \mathcal{X}$ for all $k = 1, 2, \ldots, n$.

Examples Of Basis

1. Vectors $\{\vec{i}, \vec{j}, \vec{k}\}$ form a basi s for the set of all vectors in three dimension.
2. Any three vectors
3. Any three vectors which are not coplanar form a basis in the space of vectors in three dimension.
4. $\{1, x, x^2, ..., x^N\}$ is a basis in the space of all polynomials of degree N.
5. The set $\bigcup_n \{\cos nx, \sin nx\}$, where $n = 1, 2, 3, ...$, is a basis in space of all periodic functions on $[-\pi, \pi]$ with period 2π.
6. The vectors $\mathcal{E} = e_1, e_2, ..., e_N$ where
$$e_1 = (1, 0, 0, ..., 0) \ ; e_2 = (0, 1, 0, ...0) \ ; .... \ e_N = (0, 0, 0, ..., 1)$$
form a basis in the vector space $\mathbb{C}^N$. This basis will be called the canonical basis or the standard basis.
7. The vectors $\mathcal{E} = \{e_1, e_2, ..., e_N\}$ also form a basis in R^N and in $\mathbb{Q}^N$.

Theorem 1 (Number of Elements in a Basis) *The number of elements in any one basis is equal to number of elements in every other basis.*

Definition 11 (Dimension) *For a finite dimensional space the number of elements in a basis is defined to be the dimension of the vector space.*

Summary Of Properties Of Bases

Given that a vector space $\mathcal{V}$ has dimension N we have the following properties.

1. Every set containing $N + 1$ or more vectors is a linearly dependent set.
2. A set of N vectors is a basis if and only if it is linearly independent.
3. A set of N vectors $\mathcal{X}$ is a basis iff every vector in $\mathcal{V}$ is linear combination of vectors in the set
   $\mathcal{X}$.

Definition 12 (Linear span) *Let $S = \{f_1, f_2, \ldots, f_m\}$ be subset of a vector space. The* **linear span** *of S is the set of all vectors f such that f is linear combination of vectors $f_1, f_2, \ldots, f_m \in S$. Linear Span of $S = \{f | f = \sum_{k=1}^{m} \alpha_k f_k\}$ and $f_k \in V$ and $\alpha_k \in V\}$*